A First Brief Introduction to Conformal Field Theory
(Semester Project for PC5206)

Till Dieminger
Supervised by Qinghai Wang
National University of Singapore

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Contents

Abstract

In this report we will understand the basic ideas of conformal field theory. We will first develop the notion of being conformal invariant. Afterwards we derive the algebra for different dimensions and will discover the strength of conformal field theories due to the restrictions they impose on the systems. Towards the end, we quantize the fields and find important features and methods for working with those theories. Finally we will apply those methods on an example, the free boson formulated on a world sheet. This report is mainly based on [14], [12], [5] and [10].

1 Conformal Group

Conformal Field Theory is a theory that is symmetric under the Conformal Group. This means that in a classical setting the equation of motion stays invariant under a conformal transformation, while in the quantum case, the fields of this theory carry a projective unitary representation of the Conformal Group. This part has as its goal to define, inspect and understand what conformal invarance is and means.

1.1 Conformal Mappings

We will define the notion of a conformal map in a more general setting and will later work on a more restrictive setting.

Definition. *Let $(\mathcal{M}, g_1)$ and $(\mathcal{M}, g_2)$ be two Pseudo-Riemannian Manifolds of dimension d. We call the two Riemannian metrics conformal equivalent if*

$$\exists u \in C^\infty(\mathcal{M}, \mathbb{R}^+): \ g_1 = ug_2.$$

This means, that the two metrics are rescaled at each point $p \in \mathcal{M}$. The rescaling can be different for two different points, but at a fixed point $p \in \mathcal{M}$ there exists a fixed strictly positive number u such that $g_1(a, b) = ug_2(a, b)$ for all $a, b \in T_p\mathcal{M}$

Definition. *Let $(\mathcal{M}_1, g_1)$ and $(\mathcal{M}_2, g_2)$ be two Pseudo-Riemannian Manifolds of dimension d. A diffeomorphism $\Phi \in C^\infty(U \subset \mathcal{M}_1, V \subset \mathcal{M}_2)$ is called a conformal map on U if the pullback of g_2 is conformal equivalent to g_1 on U.*

This means, that we have a local diffeomorphism Φ and a smooth function u on $U \subset \mathcal{M}_1$ such that

$$g_2|_{\Phi(p)}(d\Phi(a), d\Phi(b)) = u(p)\, g_1|_p(a, b) \quad \forall p \in U, \forall a, b \in T_p\mathcal{M}_1.$$

Definition. *We define the Conformal Group of $\mathcal{M}$ as the connected component of the set of all conformal transformations connected to the identity. As a topology we choose the standard compact open topology which is build up out of balls by the metric.*

Lemma. *The Conformal Group is a group. We denote it with $Conf(\mathcal{M}_1, \mathcal{M}_2)$*

Proof. Since all the conformal transformations are diffeomorphisms, the inverse exists and is also conformal with the conformal factor of $1/u$. Furthermore, the identity is an obvious conformal map. The composition of two conformal maps is again a conformal map with the conformal factor $u_1 \cdot u_2$. This shows that the conformal transformations form a group. Since we look at a connected subset of these transformations, they again form a subgroup. Therefore $Conf(\mathcal{M}_1, \mathcal{M}_2)$ is a group. □

Since we are interested in conformal transformation of some space to itself, we declare $\mathcal{M}_1 = \mathcal{M}_2$. We will usually work with either a Minkowski space or an Euclidean space. We denote this by $\mathbb{R}^{p,q}$, where we have the metric $\eta_{\mu\nu} = diag(-1, \dots, +1, \dots)$. We will later look at local conformal transformations of these spaces. This means, that they don't have to be defined on the whole manifold. We will always keep in mind, which transformations are globally defined. But since we want a local field theory, we will later drop the condition of the global domain in the definition of $Conf(\mathbb{R}^{p,q})$.

Example. *Let $\mathcal{M}$ be a Minkowski space. Therefore we have a global chart on $\mathcal{M}$. In this global chart, denoting the transformation as $\Phi(x) = x'$, we get the condition*

$$\eta_{\rho\sigma} \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = u(x)\eta_{\mu\nu}. \tag{1}$$

Setting the scale factor $u(x) = 1$ and requiring that the diffeomorphisms are linear, we get the definition of a Lorentz invariant transformation. This shows that Lorentz transformations are conformal maps.

Example. *Let $\mathcal{M}$ be an Euclidean space. In such a space we can define an angle by*

$$\angle a, b = \frac{g(a, b)}{\sqrt{g(a, a)g(b, b)}} \ \forall a, b \in T_p\mathcal{M}.$$

If we now make a conformal transformation each metric picks up the same factor $u(x)$. Therefore the overall factor is 1 and the angle between both tangent vectors stays invariant. Thats the reason why one can often read the statement that conformal maps preserve angles. But this notion of an angle only makes sense on a Riemannian manifold. For Pseudo-Riemannian manifolds there is no definition of an angle because the square root could be ill-defined. Therefore such a statement is not possible on pseudo Riemannian manifolds like for example the Minkowski space.

1.2 Classification of Conformal Maps in d>2

As we later have to find representations of $Conf(\mathbb{R}^{p,q})$, we proceed by firstly studying local infinitesimal coordinate transformations. A mathematical rigorous approach would be the definition of Killing fields. This introduces a lot of mathematical overhang to reach the conclusion. Because of this we will use the way we usually approach such problems in physics and demand that ε is "small" and develop a local coordinate transformation up to first order in this small parameter.

$$x'^{\rho} = x^{\rho} + \varepsilon^{\rho}(x) + \mathfrak{O}(\varepsilon^2)$$

If we now use this expansion in eq. (1) we find

$$\eta_{\mu\nu}u(x) = \eta_{\alpha\beta}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} = (\delta^{\alpha}_{\mu} - \partial_{\mu}\varepsilon^{\alpha} + \mathfrak{O}(\varepsilon^2))(\delta^{\beta}_{\nu} - \partial_{\nu}\varepsilon^{\beta} + \mathfrak{O}(\varepsilon^2))\eta_{\alpha\beta} = g_{\mu\nu} - (\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu}) + \mathfrak{O}(\varepsilon^2).$$

If we want to have an infinitesimal conformal map, we therefore require

$$\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu} = u(x)\eta_{\mu\nu} \tag{2}$$

for some strictly positive function u. We actually can determine this function by taking the trace of this equation, meaning we contract it with $\eta^{\mu\nu}$.

$$2\partial^{\mu}\varepsilon_{\mu} = \eta^{\mu\nu}(\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu}) = u(x)\eta^{\mu\nu}\eta_{\mu\nu} = u(x)(p+q) = u(x)d$$

With this we have a first restriction on the infinitesimal transformation

$$\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu} = \frac{2}{d}(\partial\cdot\varepsilon)\eta_{\mu\nu}. \tag{3}$$

It turns out that this form of the equation is not particularly useful and we therefore modify it by taking derivatives. We first take the derivative with respect to ν.

$$\partial^{\nu}(\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu}) = \frac{2}{d}\partial_{\mu}(\partial\cdot\varepsilon) \tag{4}$$

If we now introduce the notation $\partial^2 = \partial^{\mu}\partial_{\mu}$ and take a further derivative with respect to ν we find

$$\partial_{\mu}\partial_{\nu}(\partial\cdot\varepsilon) + \partial^2\partial_{\nu}\varepsilon_{\mu} = \frac{2}{d}\partial_{\mu}\partial_{\nu}(\partial\cdot\varepsilon). \tag{5}$$

If we rewrite equation eq. (5) with swapped indices $\mu \mapsto \nu \mapsto \mu$ we get

$$\partial_{\nu}\partial_{\mu}(\partial\cdot\varepsilon) + \partial^2\partial_{\mu}\varepsilon_{\nu} = \frac{2}{d}\partial_{\nu}\partial_{\mu}(\partial\cdot\varepsilon).$$

Now we can add those two equations together to find

$$2\partial_{\mu}\partial_{\nu}(\partial\cdot\varepsilon) + \partial^2\left(\frac{2}{d}(\partial\cdot\varepsilon)\eta_{\mu\nu}\right) = \frac{4}{d}\partial_{\mu}\partial_{\nu}(\partial\cdot\varepsilon).$$

Dividing by two and bringing everything to the left side we end up with

$$\left(\eta_{\mu\nu}\partial^2 + (d-2)\partial_{\mu}\partial_{\nu}\right)(\partial\cdot\varepsilon) = 0. \tag{6}$$

This is a first hint that at $d = 2$ we encounter something different than in other dimensions. Also $d = 1$ has a different form. And indeed, if we look at $d = 1$ the definition of a conformal map is met by every diffeomorphism on $\mathbb{R}$, while in other dimensions we have more restrictions.

A further formula we will use later can be found if we take the derivative of eq. (4) with respect to ρ and permute the indices. We end up with the following three equations

$$\partial_{\rho}\partial_{\mu}\varepsilon_{\nu} + \partial_{\rho}\partial_{\nu}\varepsilon_{\mu} = \frac{2}{d}\eta_{\mu\nu}\partial_{\rho}(\partial\cdot\varepsilon), \tag{7a}$$

$$\partial_{\nu}\partial_{\rho}\varepsilon_{\mu} + \partial_{\mu}\partial_{\rho}\varepsilon_{\nu} = \frac{2}{d}\eta_{\rho\mu}\partial_{\nu}(\partial\cdot\varepsilon), \tag{7b}$$

$$\partial_{\mu}\partial_{\nu}\varepsilon_{\rho} + \partial_{\nu}\partial_{\mu}\varepsilon_{\rho} = \frac{2}{d}\eta_{\nu\rho}\partial_{\mu}(\partial\cdot\varepsilon). \tag{7c}$$

If we now look at (b)+(c)-(a) we find the last equation for this section:

$$2\partial_{\mu}\partial_{\nu}\varepsilon_{\rho} = \frac{2}{d}(\eta_{\nu\rho}\partial_{\mu} + \eta_{\rho\mu}\partial_{\nu} - \eta_{\mu\nu}\partial_{\rho})(\partial\cdot\varepsilon). \tag{8}$$

We now have all the tools to classify the conformal transformations in different dimensions. As mentioned above, we have seen, that $d = 2$ needs separate consideration.

1.2.1 Conformal Transformation in d>2

If we take the trace of eq. (6), we find $(d-1)\partial^2(\partial\cdot\varepsilon) = 0$. This shows that $\partial\cdot\varepsilon$ can be at most linear in x. Therefore, ε is at most quadratic in x with a symmetric quadratic term. Motivated by this, we now can make the Ansatz

$$\varepsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho,$$

where all parameters are small and $c_{\mu\nu\rho} = c_{\mu\rho\nu}$ is symmetric. We now use the constraints on the transformation ε to find constraints on the parameters. It turns out, that we already know some of the transformations.

- Since all constraints contain derivatives, the parameter a is constraint free. It corresponds to the translation $x' = x+a$. We know that this is connected to the momentum generator. Therefore we have a first generator $P_\mu = -\mathrm{i}\partial_\mu$.
- Inserting our Ansatz up to linear order into eq. (3) we find

  $$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}tr(b)\eta_{\mu\nu}$$

  In general any operator can be split into a symmetric and an anti symmetric part. We see that the symmetric part is proportional to $\eta_{\mu\nu}$ with proportionality constant $tr(b)/d$. This can be written as $b_{\mu\nu} = tr(b)/d\,\eta_{\mu\nu} + m_{\mu\nu}$, where $m_{\mu\nu}$ is anti symmetric.

  - Assuming $m_{\mu\nu} = 0$ we find $x'^\mu = x^\mu + tr(b)/d\ x^\mu = (1+\alpha)x^\mu$. Therefore this corresponds to an infinitesimal scale transformation. We claim that the generator is given by $D = -\mathrm{i}x^\mu\partial_\mu$.
    This can be shown by making an expansion $\exp(\mathrm{i}\alpha D)x^\nu = (1 + (\mathrm{i}\alpha)(-\mathrm{i}x^\mu\partial_\mu))x^\nu = x^\nu + \alpha x^\mu\delta^\nu_\mu = x^\nu + \alpha x^\nu$.
  - Assuming $\alpha = 0$, we have a standard rotation that we already know from the Poincaré group. The known corresponding generator is $L_{\mu\nu} = \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)$.
- Setting the other parameters to zero and only looking at the quadratic term, we use eq. (8) to find

  $$\begin{aligned}
  2\partial_\mu\partial_\nu(c_{\rho\alpha\beta}x^\alpha x^\beta) &= \frac{2}{d}(\eta_{\mu\rho}\partial_\nu(\partial\cdot\varepsilon) + \eta_{\nu\rho}\partial_\mu(\partial\cdot\varepsilon) - \eta_{\mu\nu}\partial_\rho(\partial\cdot\varepsilon)) \\
  2c_{\rho\alpha\beta}\left(\delta^\alpha_\nu\delta^\beta_\mu + \delta^\alpha_\mu\delta^\beta_\nu\right) &= \frac{2}{d}(\eta_{\mu\rho}\partial_\nu(2c^\alpha{}_{\alpha\beta}x^\beta) + \eta_{\nu\rho}\partial_\mu(2c^\alpha{}_{\alpha\beta}x^\beta) - \eta_{\mu\nu}\partial_\rho(2c^\alpha{}_{\alpha\beta}x^\beta)) \\
  4c_{\rho\nu\mu} &= \frac{4}{d}(\eta_{\mu\rho}c^\alpha{}_{\alpha\nu} + \eta_{\nu\rho}c^\alpha{}_{\alpha\mu} - \eta_{\mu\nu}c^\alpha{}_{\alpha\rho})
  \end{aligned}$$

  If we now define $f_\mu = c^\alpha{}_{\alpha\mu}/d$ we can write the transformation in the following form :

  $$\begin{aligned}
  x'^\mu &= x^\mu + \varepsilon^\mu = x^\mu + c_{\mu\nu\rho}x^\nu x^\rho \\
  &= x^\mu + \eta_{\mu\rho}f_\nu x^\nu x^\rho + \eta_{\mu\nu}f_\rho x^\nu x^\rho - \eta_{\nu\rho}f_\mu x^\nu x^\rho \\
  &= x^\mu + 2x^\mu(f\cdot x) - f^\mu x^2.
  \end{aligned}$$

  We claim that the generator for this transformation is given by $K_\mu = -\mathrm{i}(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)$. That in turn can be verified by a simple calculation

  $exp(\mathrm{i}f^\mu K_\mu)x^\alpha = (1 + \mathrm{i}f^\mu(-\mathrm{i}(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)))x^\alpha = x^\alpha - f^\mu\delta^\alpha_\mu x^2 + f^\mu 2x_\mu x^\nu\delta^\alpha_\nu = x^\alpha + 2(f\cdot x)x^\alpha - f^\alpha x^2$

  We call this transformation Special Conformal Transformation and abbreviate it with SCT.

In summary we found four different types of transformations. This is a first classification of the Conformal Group. The next step is to build the algebra corresponding to these four generators. We hope that we will find the Poincaré algebra as a subalgebra since the Poincaré transformations are part of the Conformal Group.

Example. *We want to get an intuitive understanding for the SCT. For this we need to find a finite form for it, since we intuitively think in finite terms. We first claim that*

$$x'^\mu = \frac{x^\mu - f^\mu x^2}{1 - 2(f\cdot x) + f^2x^2} \tag{9}$$

is the finite form. We will proof this at the end of the example. If we now assume eq. (9), we can calculate the norm

$$x'^\mu x'_\mu = \frac{x^\mu - f^\mu x^2}{1-2f\cdot x + f^2x^2}\,\frac{x_\mu - f_\mu x^2}{1-2f\cdot x+f^2x^2} = \frac{x^2(1-2f\cdot x) + f^2x^2}{(1-2f\cdot x+f^2x^2)^2} = \frac{x^2}{1-2f\cdot x+f^2x^2}$$

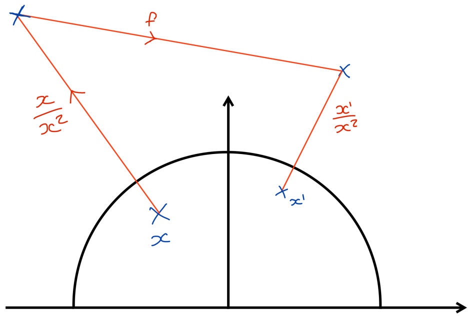

Figure 1: Visualization of the Special Conformal Transformation as the translation of the inverse coordinate.

This means that

$$\frac{x'^\mu}{x' \cdot x'} = \frac{x^\mu - f^\mu x^2}{1 - 2f \cdot x + f^2 x^2} \frac{1 - 2f \cdot x + f^2 x^2}{x^2} = \frac{x^\mu}{x \cdot x} - f^\mu.$$

This gives us an intuitive understanding of the SCT. The finite transformation is an inversion at a circle followed by a translation followed by a further inversion. Therefore this is a translation of the inverse coordinates.

To show that eq. (9) is the finite version of the SCT we show that it is a conformal transformation that is infinitesimal given by the SCT. For this we develop x' in the parameter f.

$$x'^\mu = (x^\mu - f^\mu x^2)(1 - 2f \cdot x + f^2 x^2)^{-1} \approx (x^\mu - f^\mu x^2)(1 + 2x \cdot f + 3x^2 f^2) \approx x^\mu + 2(f \cdot x)x^\mu - f^\mu x^2$$

To see that the transformation is conformal we calculate the derivative.

$$\begin{aligned}
\frac{\partial x'^\mu}{\partial x^\sigma} &= \frac{\partial_\sigma (x^\mu - f^\mu x^2)}{1 - 2(f \cdot x) + f^2 x^2} + (x^\mu - f^\mu x^2)\partial_\sigma (1 - 2(f \cdot x) + f^2 x^2)^{-1} \\
&= \frac{\delta^\mu_\sigma - f^\mu(\delta^\alpha_\sigma x_\alpha + s^\alpha \delta^\sigma_\alpha)}{1 - 2(f \cdot x) + f^2 x^2} - \frac{\partial_\sigma (1 - 2(f \cdot x) + f^2 x^2)}{(1 - 2(f \cdot x) + f^2 x^2)^2} \\
&= \frac{\delta^\mu_\sigma - 2f \cdot x\delta^\mu_\sigma + b^2 x^2 \delta^\mu_\sigma - 2f^\mu x_\sigma + 4(f \cdot x) f^\mu x_\sigma - 2f^2 x^2 f^\mu x_\sigma + 2x^\mu f_\sigma - 2f_\sigma f^\mu x^2 - 2f^2 x^\mu x_\sigma + f^\mu x^2 2f^2 x_\sigma}{(1 - 2(f \cdot x) + f^2 x^2)^2} \\
&= \frac{\delta^\mu_\sigma (1 - 2(f \cdot x) + f^2 x^2)}{(1 - 2(f \cdot x) + f^2 x^2)^2} \\
&= \frac{\delta^\mu_\sigma}{1 - 2(f \cdot x) + f^2 x^2}
\end{aligned}$$

Since the derivative is proportional to δ^μ_σ, this map is conformal. We further can determine the conformal factor

$$u(x) = \frac{\partial x^\sigma}{\partial x'^\sigma} \frac{\partial x^\rho}{\partial x'^\nu} = (1 - 2(f \cdot x) + f^2 x^2)^2.$$

In summary we have shown that the SCT can be integrated to equation 9 which can be understood as a translation of the inverse coordinates.

In summary we found the following conformal transformations:

Transformation	Finite Form	Generator
Translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
Dilation	$x'^\mu = \alpha x^\mu$	$D = -ix^\mu \partial_\mu$
Rotation	$x'^\mu = M^\mu{}_\nu x^\nu$	$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
SCT	$x'^\mu = \frac{x^\mu - f^\mu x^2}{1 - 2(f \cdot x) + f^2 x^2}$	$K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

Figure 2: The conformal transformations with the corresponding generators.

1.2.2 Conformal Algebra in d>2

To build up the algebra, we first calculate all the commutators.

$$\left[D,P_\mu\right] = -x^\alpha\partial_\alpha\partial_\mu + \partial_\mu(x^\alpha\partial_\alpha) = \delta^\alpha_\mu\partial_\alpha = \mathrm{i}P_\mu$$

$$\begin{aligned}
\left[D,K_\mu\right] &= -\mathrm{i}(x^\alpha\partial_\alpha)(-\mathrm{i}(2x_\mu x^\beta\partial_\beta - x^2\partial_\mu)) - (-\mathrm{i}(2x_\mu x^\beta\partial_\beta - x^2\partial_\mu))(-\mathrm{i}(x^\alpha\partial_\alpha)) \\
&= -x^\alpha\partial_\alpha(2x_\mu x^\beta\partial_\beta) + x^\alpha\partial_\alpha(x^2\partial_\mu) + 2x_\mu x^\gamma\partial_\gamma(x\cdot\partial) - x^2\partial_\mu(x\cdot\partial) \\
&= -2\eta_{\gamma\mu}x^\alpha\delta^\gamma_\alpha x^\beta\partial_\beta - 2\eta_{\gamma\mu}x^\alpha x^\gamma\delta^\beta_\alpha\partial_\beta - 2x^\alpha x^\mu\partial_\alpha\partial_\beta + x^\alpha\delta^\gamma_\alpha\eta_{\gamma\varepsilon}x^\varepsilon\partial_\mu + x^\alpha x^\gamma\eta_{\gamma\varepsilon}\delta^\varepsilon_\alpha\partial_\mu \\
&\quad + x^\alpha x^2\partial_\alpha\partial_\mu + 2x_\mu(x\cdot\partial) + 2x_\mu x^\gamma x^\varepsilon\partial_\gamma\partial_\varepsilon - x^2\partial_\mu - x^2(x\cdot\partial)\partial_\mu \\
&= -2x_\mu(x\cdot\partial) + x^2\partial_\mu = -\mathrm{i}K_\mu
\end{aligned}$$

$$\begin{aligned}
\left[K_\mu,P_\nu\right] &= (-\mathrm{i}(2x_\mu(x\cdot\partial) - x^2\partial_\mu)(-\mathrm{i}\partial_\nu)) - (-\mathrm{i}\partial_\nu)(-\mathrm{i}(2x_\mu(x\cdot\partial) - x^2\partial_\mu)) \\
&= -2x_\mu(x\cdot\partial)\partial_\nu + x^2\partial_\mu\partial_\nu + 2\partial_\nu(\eta_{\alpha\mu}x^\alpha(x\cdot\partial - \partial_\nu(x^2\partial_\mu))) \\
&= -2x_\mu(x\cdot\partial)\partial_\nu + x^2\partial_\mu\partial_\nu + 2\eta_{\nu\mu}(x\cdot\partial) + 2x_\mu\delta^\alpha_\nu\partial_\alpha \\
&\quad + 2x_\mu x^\alpha\partial_\alpha\partial_\nu - \delta^\alpha_\nu\eta_{\alpha\beta}x^\beta\partial_\mu - x^\alpha\eta_{\alpha\beta}\delta^\beta_\nu\partial_\mu - x^2\partial_\mu\partial_\nu \\
&= 2(\eta_{\mu\nu}(x\cdot\partial) - x_\mu\partial_\nu + x_\nu\partial_\mu) \\
&= 2\mathrm{i}(\eta_{\mu\nu}D - L_{\mu\nu})
\end{aligned}$$

$$\begin{aligned}
\left[K_\rho,L_{\mu\nu}\right] &= -\mathrm{i}(2x_\rho(x\cdot\partial) - x^2\partial_\rho)\mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu) - \mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)(-\mathrm{i}(2x_\rho(x\cdot\partial) - x^2\partial_\rho)) \\
&= 2x_\rho(x\cdot\partial)x_\mu\partial_\nu - 2x_\rho(x\cdot\partial)x_\nu\partial_\mu - x^2\partial_\rho(x_\mu\partial_\nu) + x^2\partial_\rho(x_\nu\partial_\mu) \\
&\quad - x_\mu\partial_\nu(2x_\rho(x\cdot\partial)) + x_\mu\partial_\nu(x^2\partial_\rho) +_\nu \partial_\mu(2x_\rho(x\cdot\partial)) - x_\nu\partial_\mu(x^2\partial_\rho) \\
&= 2x_\rho x_\varepsilon(\delta^\varepsilon_\mu\partial_\nu + x_\mu\partial^\varepsilon\partial_\nu) - 2x_\rho x_\varepsilon(\delta^\varepsilon_\nu\partial_\mu + x_\nu\partial^\varepsilon\partial_\mu) - x^2(\eta_{\rho\mu}\partial_\nu + x_\mu\partial_\rho\partial_\nu) \\
&\quad + x^2(\eta_{\rho\nu}\partial_\mu + x_\nu\partial_\rho\partial_\mu) - 2x_\mu(\eta_{\nu\rho}(x\cdot\partial) + x_\rho\delta^\varepsilon_\nu\partial_\varepsilon + x_\rho(x\cdot\partial)\partial_\nu) \\
&\quad + x_\mu\eta_{\varepsilon\gamma}(\delta^\varepsilon_\nu x^\gamma\partial_\rho + x^\varepsilon\delta^\gamma_\nu\partial_\rho + x^\varepsilon x^\gamma\partial_\rho\partial_\nu) \\
&\quad + x_\nu(\eta_{\mu\rho}(x\cdot\partial) + x_\rho\delta^\gamma_\mu\partial_\gamma x_\rho(x\cdot\partial)\partial_\mu) - x_\nu\eta_{\gamma\varepsilon}(\delta^\varepsilon_\mu x^\gamma\partial_\rho + \delta^\gamma_\mu x^\varepsilon\partial_\rho + x^\varepsilon x^\gamma\partial_\mu\partial_\nu) \\
&= x^2(\eta_{\rho\nu}\partial_\mu - \eta_{\rho\mu}\partial_\nu) + 2(x_\nu\eta_{\mu\rho} - x_\mu\eta_{\nu\rho})(x\cdot\partial) \\
&= 2(\eta_{\mu\rho}K_\nu - \eta_{\rho\nu}K_\mu)
\end{aligned}$$

$$\begin{aligned}
\left[P_\rho,L_{\mu\nu}\right] &= (-\mathrm{i}\partial_\rho)(\mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu)) - (\mathrm{i}(x_\mu\partial_\nu - x_\nu\partial_\mu))(-\mathrm{i}\partial_\rho) \\
&= \partial_\rho(x_\mu\partial_\nu) - \partial_\rho(x_\nu\partial_\mu) - (x_\mu\partial_\nu\partial_\rho) + x_\nu\partial_\mu\partial_\rho \\
&= \eta_{\mu\rho}\partial_\rho + x_\mu\partial_\rho\partial_\nu - \eta_{\rho\nu}\partial_\mu - x_\nu\partial_\rho\partial_\mu - x_\mu\partial_\nu\partial_\rho + x_\nu\partial_\mu\partial_\rho \\
&= \eta_{\rho\mu}\partial_\nu - \eta_{\rho\nu}\partial_\mu \\
&= \mathrm{i}(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)
\end{aligned}$$

$$\begin{aligned}
\left[K_\mu,K_\nu\right] &= -\mathrm{i}(2x_\mu(x\cdot\partial) - x^2\partial_\mu)(-\mathrm{i})(2x_\nu(x\cdot\partial) - x^2\partial_\nu) - (-\mathrm{i}(2x_\mu(x\cdot\partial) - x^2\partial_\mu))(-\mathrm{i})(2x_\nu(x\cdot\partial) - x^2\partial_\nu) \\
&= -\mathrm{i}(4x_\mu x_\alpha\partial^\alpha(x_\nu(x\cdot\partial)) - 2x_\mu x^\alpha\partial_\alpha x^\gamma x^\varepsilon\eta_{\varepsilon\gamma}\partial_\nu) - 2x^2\partial_\mu(x_\nu x^\alpha\partial_\alpha) \\
&\quad - 4x_\nu x_\gamma\partial^\gamma(x_\mu x^\varepsilon\partial_\varepsilon) - 2x_\nu x^\gamma\partial_\gamma(x^2\partial_\mu) - x^2\partial_\nu(2x_\mu x^\alpha\partial_\alpha) \\
&= -2x_\mu x^2\partial_\nu - 2x_\mu x^2\partial_\nu - 2x^2x_\mu(x\cdot\partial)\partial_\nu - 2x^2x_\nu\partial_\mu - 2x^2x_\nu(x\cdot\partial)\partial_\mu \\
&\quad + 2x_\mu x^2\partial_\nu + 2x_\nu x^2\partial_\mu + 2x^2x_\nu(x\cdot\partial)\partial_\mu + 2x^2x_\mu\partial_\nu + 2x^2x_\mu(x\cdot\partial)\partial_\nu = 0
\end{aligned}$$

$$\left[P_\mu,P_\nu\right] = -\mathrm{i}\partial_\mu(-\mathrm{i}\partial_\nu) - (-\mathrm{i}\partial_\nu)(-\mathrm{i}\partial_\mu) = -\left[\partial_\mu,\partial_\nu\right] = 0$$

$$[D,D] = 0$$

We further have the well known commutator of the angular momentum generators given by

$$\left[L_{\mu\nu}, L_{\rho\sigma}\right] = \mathrm{i}(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}).$$

In summary we have the following commutation relations

$$\begin{aligned}
&\left[D, P_\mu\right] = \mathrm{i}P_\mu, \\
&\left[D, K_\mu\right] = -\mathrm{i}K_\mu, \\
&\left[K_\mu, P_\nu\right] = 2\mathrm{i}(\eta_{\mu\nu}D - L_{\mu\nu}), \\
&\left[K_\rho, L_{\mu\nu}\right] = 2(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu), \\
&\left[P_\rho, L_{\mu\nu}\right] = 2(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \\
&\left[L_{\mu\nu}, L_{\rho\sigma}\right] = \mathrm{i}(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}), \\
&\left[P_\mu, P_\nu\right] = \left[K_\mu, K_\nu\right] = [D, D] = 0.
\end{aligned}$$

Considering the number of different generators and keeping in mind that $L_{\mu\nu}$ is anti symmetric we get the dimension of the conformal algebra $N = d + 1 + \frac{d(d-1)}{2} + d = \frac{(d+2)(d+1)}{2}$. We now can define a generator

$$J_{\mu\nu} = L_{\mu\nu}, \qquad J_{-1,0} = D, \qquad J_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu), \qquad J_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu), \qquad J_{mn} = -J_{nm}$$

which combines the commutation relations into one equation. Let $m, n, r, s \in \{-1, 0, ..., (d-1)\}$, we claim that

$$[J_{mn}, J_{rs}] = \mathrm{i}(\eta_{ms}J_{nr} + \eta_{nr}J_{ms} - \eta_{mr}J_{ns} - \eta_{ns}J_{mr}). \tag{10}$$

Here η is the metric of $\mathbb{R}^{1,d+1}$. To summarize, we found that the conformal algebra is isomorphic to $\mathfrak{so}(d+1, 1)$.

We see that eq. (10) holds for $m, n = \mu, \nu$ since this is just the commutation relation of the angular momentum. For $[J_{-1,0}, J_{-1,0}] = [D, D] = 0$ we again have trivial commutation relations. We see that $P_\mu = J_{-1,\mu} + J_{0,\mu}$ and $K_\mu = J_{0,\mu} - J_{-1,\mu}$. Further the asymmetry condition of J implies that the diagonal elements vanish. Using these relations, one can calculate the commutation relations and show that eq. (10) is really equivalent to the conformal algebra.

$$\begin{aligned}
\left[D, P_\mu\right] &= \left[J_{-1,0}, J_{-1,\mu} + J_{0,\mu}\right] \\
&= \left[J_{-1,0}, J_{-1,\mu}\right] + \left[J_{-1,0}, J_{0,\mu}\right] \\
&= \mathrm{i}(\eta_{-1,0}J_{0,-1} + \eta_{0,-1}J_{-1,\mu} - \eta_{-1,-1}J_{0,\mu} - \eta_{0,\mu}J_{-1,-1}) \\
&\quad + \mathrm{i}(\eta_{0,0}J_{-1,\mu} + \eta_{-1,\mu}J_{0,0} - \eta_{-1,0}J_{0,\mu} - \eta_{0,\mu}J_{-1,0}) \\
&= \mathrm{i}(J_{0,\mu} + J_{-1,\mu}) = \mathrm{i}P_\mu
\end{aligned}$$

$$\begin{aligned}
\left[D, K_\mu\right] &= \left[J_{-1,0}, J_{0,\mu} - J_{-1,\mu}\right] \\
&= \left[J_{-1,0}, J_{0,\mu}\right] - \left[J_{-1,0}, J_{-1,\mu}\right] \\
&= \mathrm{i}(\eta_{0,0}J_{-1,\mu} + \eta_{-1,\mu}J_{0,0} - \eta_{-1,0}J_{0,\mu} - \eta_{0,\mu}J_{-1,0}) \\
&\quad - \mathrm{i}(\eta_{-1,0}J_{0,-1} + \eta_{0,-1}J_{-1,\mu} - \eta_{-1,-1}J_{0,\mu} - \eta_{0,\mu}J_{-1,-1}) \\
&= \mathrm{i}(J_{-1,\mu} - J_{0,\mu}) = -\mathrm{i}K_\mu
\end{aligned}$$

$$\begin{aligned}
\left[K_\mu, P_\nu\right] &= \left[J_{0,\mu} - J_{-1,\mu}, J_{-1,\nu} + J_{0,\nu}\right] \\
&= \left[J_{0,\mu}, J_{-1,\nu}\right] + \left[J_{0,\mu}, J_{0,\nu}\right] - \left[J_{-1,\mu}, J_{-1,\nu}\right] - \left[J_{-1,\mu}, J_{0,\nu}\right] \\
&= \mathrm{i}(\eta_{0,\nu}J_{\mu,-1} + \eta_{\mu,-1}J_{0,\nu} - \eta_{0,-1}J_{\mu,\nu} - \eta_{\mu,\nu}J_{0,-1}) \\
&\quad + \mathrm{i}(\eta_{0,\nu}J_{\mu,0} + \eta_{\mu,0}J_{0,\nu} - \eta_{0,0}J_{\mu,\nu} - \eta_{\mu,\nu}J_{0,0}) \\
&\quad - \mathrm{i}(\eta_{-1,\nu}J_{\mu,0} + \eta_{\mu,-1}J_{-1,\nu} - \eta_{-1,-1}J_{\mu,\nu} - \eta_{\mu,\nu}J_{-1,-1}) \\
&\quad - \mathrm{i}(\eta_{-1,\nu}J_{\mu,0} + \eta_{\mu,0}J_{-1,\nu} - \eta_{-1,0}J_{\mu,\nu} - \eta_{\mu,\nu}J_{-1,0}) \\
&= \mathrm{i}(\eta_{\mu,\nu}D - \eta_{\mu,\nu}(-D) - J_{\mu,\nu} - J_{\mu,\nu}) \\
&= 2\mathrm{i}(\eta_{\mu\nu}D - L_{\mu\nu})
\end{aligned}$$

$$
\begin{aligned}
\left[K_\rho, L_{\mu\nu}\right] &= \left[J_{0,\rho} - J_{-1,\rho}, J_{\mu,\nu}\right] \\
&= \left[J_{0,\rho}, J_{\mu,\nu}\right] - \left[J_{-1,\rho}, J_{\mu,\nu}\right] \\
&= \mathrm{i}(\eta_{0,\nu} J_{\rho,\mu} + \eta_{\rho,\mu} J_{-,\nu} - \eta_{0,\mu} J_{\rho,\nu} - \eta_{\rho,\nu} J_{0,\mu}) \\
&- \mathrm{i}(\eta_{-1,\nu} J_{\rho,\mu} + \eta_{\rho,\mu} J_{-1,\nu} - \eta_{-1,\mu} J_{\rho,\nu} - \eta_{\rho,\nu} J_{-1,\mu}) \\
&= \mathrm{i}(\eta_{\rho,\mu}(J_{0,\nu} - J_{-1,\nu}) - \eta_{\rho,\nu}(J_{0,\mu} - J_{-1,\mu})) \\
&= 2\mathrm{i}(\eta_{\rho,\mu} K_\nu - \eta_{\rho,\nu} K_\mu)
\end{aligned}
$$

$$
\begin{aligned}
\left[P_\rho, L_{\mu\nu}\right] &= \left[J_{0,\rho} + J_{-1,\rho}, J_{\mu,\nu}\right] \\
&= \left[J_{0,\rho}, J_{\mu,\nu}\right] + \left[J_{-1,\rho}, J_{\mu,\nu}\right] \\
&= \mathrm{i}(\eta_{0,\nu} J_{\rho,\mu} + \eta_{\rho,\mu} J_{-,\nu} - \eta_{0,\mu} J_{\rho,\nu} - \eta_{\rho,\nu} J_{0,\mu}) \\
&\quad + \mathrm{i}(\eta_{-1,\nu} J_{\rho,\mu} + \eta_{\rho,\mu} J_{-1,\nu} - \eta_{-1,\mu} J_{\rho,\nu} - \eta_{\rho,\nu} J_{-1,\mu}) \\
&= \mathrm{i}(\eta_{\rho,\mu}(J_{0,\nu} + J_{-1,\nu}) - \eta_{\rho,\nu}(J_{0,\mu} - J_{-1,\mu})) \\
&= 2\mathrm{i}(\eta_{\rho,\mu} P_\nu - \eta_{\rho,\nu} P_\mu)
\end{aligned}
$$

$$
\begin{aligned}
\left[P_\mu, P_\nu\right] &= \left[J_{0,\mu} + J_{-1,\mu}, J_{0,\nu} + J_{-1,\nu}\right] \\
&= \mathrm{i}(\eta_{0,\nu} J_{\mu,0} + \eta_{\mu,0} J_{0,\nu} - \eta_{0,0} J_{\mu,\nu} - \eta_{\mu,\nu} J_{0,0}) \\
&\quad + \mathrm{i}(\eta_{0,\nu} J_{\mu,-1} + \eta_{\mu,-1} J_{0,\nu} - \eta_{0,-1} J_{\mu,\nu} - \eta_{\mu,\nu} J_{0,-1}) \\
&\quad + \mathrm{i}(\eta_{-1,\nu} J_{\mu,0} + \eta_{\mu,0} J_{-1,\nu} - \eta_{-1,0} J_{\mu,\nu} - \eta_{\mu,\nu} J_{-1,0}) \\
&\quad + \mathrm{i}(\eta_{-1,\nu} J_{\mu,-1} + \eta_{\mu,-1} J_{-1,\nu} - \eta_{-1,-1} J_{\mu,\nu} - \eta_{\mu,\nu} J_{-1,-1}) \\
&= \mathrm{i}(-\eta_{0,0} J_{\mu,\nu} - \eta_{\mu,\nu}(J_{0,-1} + J_{-1,0}) - \eta_{-1,-1} J_{\mu,\nu}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\left[K_\mu, K_\nu\right] &= \mathrm{i}\left[J_{0,\mu} - J_{-1,\mu}, J_{0,\nu} - J_{-1,\nu}\right] \\
&= \mathrm{i}\left[J_{0,\mu}, J_{0,\nu}\right] + \left[J_{-1,\mu}, J_{-1,\nu}\right] - \left[J_{-1,\mu}, J_{0,\nu}\right] - \left[J_{0,\mu}, J_{-1,\nu}\right] \\
&= -\eta_{0,0} J_{\mu,\nu} - \eta_{-1,-1} J_{\mu,\nu} + \eta_{\mu,\nu} J_{-1,0} + \eta_{\mu,\nu} J_{0,-1} \\
&= -(\eta_{0,0} + \eta_{-1,-1}) J_{\mu,\nu} \\
&= 0
\end{aligned}
$$

This isomorphism is the key idea of the so called embedding space formalism. Here one uses an embedding and defines actions in $\mathbb{R}^{1,d+1}$ instead of $\mathbb{R}^d$.

1.2.3 Conformal Representation for d>2

To build the representation theory for the conformal group, we use what we already know from the Poincaré representation. Let $\psi(0)$ be a irreducible representation of the rotation group SO(d). This means we have

$$L_{\mu\nu}\psi(0) = (\mathcal{S}_{\mu\nu})\psi(0).$$

To find the action of L at an arbitrary point on the space we have to use the Campbell-Baker-Hausdorff formula to get

$$
\begin{aligned}
L_{\mu\nu}\psi(x) &= L_{\mu\nu} \exp(\mathrm{i}x \cdot P) \\
&= \exp(\mathrm{i}x \cdot P)(exp(-\mathrm{i}x \cdot P) L_{\mu\nu} exp(\mathrm{i}x \cdot P))\psi(0) \\
&= exp(\mathrm{i}x \cdot P)(m_{\mu\nu} + \mathcal{S}_{\mu\nu})\psi(0) \\
&= (m_{\mu\nu} + \mathcal{S}_{\mu\nu})\psi(x).
\end{aligned}
$$

Because we have a scale invariant theory, we are tempted to diagonalize the scaling operator D. This means we have a real value h such that

$$D\psi(0) = \mathrm{i}h\psi(0).$$

This can be further justified as we assume to work in an irreducible representation of the Pioncaré algebra. By Schur's Lemma and because D commutes with $L_{\mu\nu}$, D must be a multiple of the identity. We can now do the same steps as for the Poincaré transformation to get $D\psi(x)$.

$$\begin{aligned}
D\psi(x) &= De^{\mathrm{i}x\cdot P}\psi(0) \\
&= e^{\mathrm{i}x\cdot P}(e^{-\mathrm{i}x\cdot P}De^{\mathrm{i}x\cdot P})\psi(0) \\
&= e^{\mathrm{i}x\cdot P}(D - [\mathrm{i}x\cdot P, D]) \\
&= h\psi(x) - \mathrm{ii}e^{\mathrm{i}x\cdot P}x\cdot P\psi(0) \\
&= (h + x\cdot\partial)\psi(x)
\end{aligned}$$

If we now include the SCT in our algebra, we can find a lowering and raising operator with respect to D.

$$DK_\mu\psi(0) = \left(\left[D, K_\mu\right] + K_\mu D\right)\psi(0) = (-\mathrm{i} + \mathrm{i}h)K_\mu\psi(0)$$

Looking at the same equation with P_μ we find a raising operator.

$$DP_\mu\psi(0) = \left(\left[D, P_\mu\right] + P_\mu\right)\psi(0) = (\mathrm{i} + \mathrm{i}h)P_\mu\psi(0)$$

If we have a way to claim that the eigenvalues of the dilation are bounded from below, we would have a complete description of the representation theory of $Conf(R^{p,q})$ in d>2. We will later see that the two point correlators have the form

$$\langle\psi_1(x)\psi_2(y)\rangle = \frac{C}{|x-y|^{h_1+h_2}}.$$

Therefore, we expect the eigenvalues to be positive in order to avoid correlations that grow over distance. This argument gives the lower bound we need. In summary we have a full description of the representations similar to the case of $\int\sqcap(N)$. To complete the discussion, we should derive how K_μ acts on fields besides its lowering action. Taking a look at the algebra we see that $\left[D, K_\mu\right] = -\mathrm{i}K_\mu$. As D is diagonal, $K_\mu = 0$ at the origin. To find the action for an arbitrary point we again use the Baker Cambell Hausdorff formula to find

$$\begin{aligned}
e^{-\mathrm{i}x\cdot P}K_\mu e^{\mathrm{i}x\cdot P} &= K_\mu + \mathrm{i}x^\nu\left[P_\nu, K_\mu\right] - \frac{1}{2}x^\nu x^\rho\left[P_\rho, \left[P_\nu, K_\mu\right]\right] \\
&= K_\mu + 2x^\nu(\eta_{\mu\nu}D - L_{\mu\nu}) + x^\nu x^\rho\eta_{\mu\nu}P_\rho - x^\nu x^\rho(\mathrm{i})^2(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
&= 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu xD - x^2P_\mu
\end{aligned}$$

If a field transforms in the lowest weight state of D we call it a primary field, while, if it transforms in one of the higher weight states, we call it a descendant. We will encounter these terms again in the two dimensional case.

This closes the discussion for the higher dimensional case. We have seen that the representations of $Conf(\mathbb{R}^{p,q})$ can be built in a similar fashion to the representations of $SU(2)$ and that the group itself consists out of 4 classes of transformations and furthermore has the Poincaré algebra as a subalgebra.

1.3 Classification of Conformal Maps in d=2

We again start with an infinitesimal local transformation. In the previous part we found eq. (4) which in the two dimensional case is

$$\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu = (\partial\cdot\varepsilon)\eta_{\mu\nu}.$$

Plugging in $\mu, \nu \in 1, 2$ with an Euclidean metric we find the conditions for the two dimensional case.

$$\begin{aligned}
&\partial_1\varepsilon_2 + \partial_2\varepsilon_1 = 0, \\
&\partial_1\varepsilon_1 + \partial_1\varepsilon_1 = \partial_1\varepsilon_1 + \partial_2\varepsilon_2.
\end{aligned}$$

If we consider $\varepsilon_1, \varepsilon_2$ as two parts of a complex function, these equations are the well known Cauchy-Riemann equations which characterize the holomorphic functions. Since we work on a two dimensional space, its pretty appealing to introduce a complexified version of all the equations, because then we have a simple classification for conformal transformations, just the holomorphic functions.

$$z = x^0 + \mathrm{i}x^1, \quad \bar{z} = x^0 - \mathrm{i}x^1, \quad \varepsilon = \varepsilon^0 + \mathrm{i}\varepsilon^1, \quad \partial_z = \frac{1}{2}(\partial_0 - \mathrm{i}\partial_1)$$

Under these circumstances any holomorphic function ε gives rise to a local conformal transformation $f(z) = z + \varepsilon(z)$. Under such a tranformation we have

$$ds^2 = dzd\bar{z} \mapsto \frac{\partial f}{\partial z}\frac{\partial\bar{f}}{\partial\bar{z}}dzd\bar{z}.$$

This means that the conformal factor is given by $\left|\frac{\partial f}{\partial z}\right|^2$.

1.3.1 Conformal Algebra in d=2

To determine the algebra in the two dimensional case, we use the Laurent expansion for holomorphic functions. So we can write

$$z' = z + \sum_{\mathbb{Z}} \varepsilon_n(-z^{n+1}).$$

Here the sign is a convention. In order to find the generators, we use this transformation on a field Φ and expand it up to the first order.

$$\Phi(z') = \Phi(z - \varepsilon_n z^{n+1}) = \Phi(z) + \partial_z\Phi(z)\cdot(z'-z) = \Phi(z) - \varepsilon_n z^{n+1}\partial_z\Phi$$

This implies that the generators are given by

$$\ell_n = -z^{n+1}\partial_z$$

Calculating the commutators to determine the algebra we find

$$[\ell_m, \ell_n] = z^{m+1}\partial_z(x^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) = (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z = (m-n)\ell_{m+n}. \tag{11}$$

This algebra is is called the Witt algebra.

But we also could use the anti holomorphic transformation $\bar{z} \mapsto \bar{z} + \sum_{\mathbb{Z}} -\bar{\varepsilon}_n\bar{z}^{n+1}$. This results in the same algebra where we just put a bar over everything. We further treat z and $\bar{z}$ as independent variables, so $[\ell_n, \bar{\ell}_m] = 0$. This shows that the conformal algebra in two dimensions is the direct sum of two copies of the Witt algebra.

$$\mathfrak{conf}(\mathbb{R}^2) = Witt \oplus Witt$$

Here we take a short break to understand what we just observed. Since the Laurent expansion has infinitely many terms, we have infinitely many generators. Accordingly our algebra is infinitely dimensional. Using Noethers theorem we have infinitely many conserved quantities, allowing us to reduce the degrees of freedom to zero and therefore solving problems exactly. This is what makes the two dimensional conformal field theory so appealing and special. But may the requirements for such a system be so demanding that we won't find any non trivial systems obeying a conformal symmetry? It turns out, as we will see later, that we can build conformal invariant non trivial systems. especially String Theory will supply us with many examples of invariant systems. Therefore the requirements are not too constraining.

We just built up the infinitesimal algebra $\{\ell_n\}$ on $\mathbb{C} \simeq \mathbb{R}^2$, but are now interested in what transformations can be integrated into globally defined transformations. We already see that most of them can't be integrated, since they are not defined everywhere. Especially at $z = 0$ we have some ambiguities. Trying to fix this problem, we compactify $\mathbb{C}$ resulting in the so called Riemann sphere $\mathbb{C} \cup \{\infty\}$. But even now, almost all generators are not well defined in some region, either at $z \to 0$ or $z \to \infty$. If we want the generator to be defined globally we require them to be non singular in those regions.

- For $z \to 0$ we need to make sure that the prefactor z^{n+1} does not diverge.

$$\ell_n = -z^{n+1}\partial_z \quad \Rightarrow \quad n \geq -1$$

- For $z \to \infty$ we make a change of variable $z = -\frac{1}{w}$ and study $w \to 0$. Under this transformation we get $\ell_n = -(-w)^{-n+1}\partial_w$.

$$\ell_n = -(-\frac{1}{w})^{n-1}\partial_w \quad \Rightarrow \quad n \leq 1$$

So we see that only the three generators $\ell_{-1}, \ell_0, \ell_1$ are well defined globally and therefore have a chance to get integrated up to a global transformation. We further need to check if those three generators form an subalgebra, otherwise there is no group that could represent the global transformations. Making use of the commutations relations eq. (11) we find

$$[\ell_0, \ell_1] = -\ell_1, \qquad [\ell_0, \ell_{-1}] = \ell_{-1}, \qquad [\ell_1, \ell_{-1}] = 2\ell_0.$$

So they indeed form a subalgebra. We could ask why we only found three generators instead of the 4 we found in higher dimensions. If we recall, we arrived at four generators in higher dimensions since we could split up the linear term in our Ansatz into two different transformations, one asymmetrical and one symmetrical. The same happens in two dimensions, where one generator will actually split up into two.

If we now want to find the conformal group, we need to find the finite transformations to which these generators belong. This has the side effect that we further gain some intuition for these generators.

- Looking at the definition of ℓ_1 this is just $-\partial_z$. From the previous section we know that this belongs to the finite translation $z \mapsto z + a$.
- The second generator is given by $\ell_0 = -z\partial_z$. We claim that this generates tranformations of the form $z \mapsto az$ where $a \in \mathbb{C}$. If this is the case, we have the previous mentioned splitting into two actions. To see this one writes z in polar coordinates. We now see $\ell_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\varphi$. This shows that the radial part of a acts as a dilation while the angular parts end up rotating the phase of z. We therefore end up with two transformations just as in the higher dimensional case.
- For the last operator ℓ_1 we hope to find the SCT to have a correspondence between two and higher dimensions. If we expand the SCT we find

  $$z \mapsto \frac{z}{cz+1} \approx z - cz^2 + c^2z^3 = z - cz^2 = z - c\ell_1 z$$

  and therefore we really have found the two dimensional version of the SCT in the subalgebra.

In summary, we found an infinite dimensional algebra of infinitesimal conformal transformations and a smaller finite subalgebra. This finite subalgebra can be integrated to form the globally defined finite conformal transformations. We have seen that the two dimensional case is not so different from the higher dimensional case if we only look at globally defined transformations. If we work with local transformations on the other hand, the situation changes completely because of the size of the algebra.

1.4 Virasoro Algebra

Since we want to look at unitary protective representations, we should determine the so called second cohomology group $H^2(Witt) \simeq Witt \oplus \mathbb{C}$, as we know gives a characterization of these representations. Lets denote $\mathfrak{g} = Witt$ and $\tilde{\mathfrak{g}} = Witt \oplus \mathbb{C}$. The cohomology commutation relations are given by

$$[\tilde{x}, \tilde{y}]_{\tilde{\mathfrak{g}}} = [x, y]_{\mathfrak{g}} + cp(x, y), \quad [\tilde{x}, z]_{\tilde{\mathfrak{g}}} = [z, w]_{\tilde{\mathfrak{g}}} = 0 \quad \forall \tilde{x}, \tilde{y} \in \tilde{\mathfrak{g}}, x, y \in \mathfrak{g}, z, w \in \mathbb{C},$$

where $p(x, y)$ is a bilinear c-valued function. Let the extension of ℓ_n be given by L_n. We now want to find the function p. For this we first write down the commutator and then try to find restrictions.

$$[L_m, L_n] = (m - n)L_{m+n} + cp(m, n)$$

We know that the commutator, as it is a Lie bracket is anti symmetric. Because the first term is antisymmetric, we need that $p(m, n) = -p(n, m)$. Using a redefinition of the generators we can achieve that $p(n, 0) = p(1, -1) = 0$. We firstly redefine

$$\hat{L}_n = L_n \frac{cp(n, 0)}{n} \text{ for } n \neq 0 \quad \hat{L}_0 = L_0 + \frac{cp(1, -1)}{2}$$

and now plug these into the commutator.

$$[\hat{L}_n, \hat{L}_0] = [L_n, L_0] + [L_n, cp(1, -1)/2] + [cp(n, 0)/n, L_0] + [cp(n, 0)/n, cp(1, -1)] + cp(n, 0) = nL_n + cp(n, 0) = n\hat{L}_n$$
$$[\hat{L}_1, \hat{L}_{-1}] = 2L_0 + cp(1, -1) = 2\hat{L}_0$$

Due to this redefinition p(1,-1)=p(n,0)=0 for the new generators. Lets now omit the hats and write L_n for the redefined generator. Since we have a commutator it also needs to fulfill the Jacobi identity. We can use that to show that $p(n, m) \propto \delta_{m,-n}$.

$$\begin{aligned} 0 &= [[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] \\ &= (m - n)cp(m + n, 0) + ncp(n, m) - mcp(m, n) \\ &= (m + n)cp(n, m). \end{aligned}$$

This shows, that if we are not in the trivial representation (c=0) we have p(n,m)=0 if $m \neq -n$. We can use a further Jacobi identity to find a recursion relation for $p(n, m)$.

$$\begin{aligned} 0 &= [[L_{-n+1}, L_n]L_{-1}] + [[L_n, L_{-1}]L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] \\ &= (-2n + 1)cp(1, -1) + (n + 1)cp(n - 1, -n + 1) + (n - 2)cp(-n, n) \end{aligned}$$

Since the first term vanishes after our redefinition, we find the simple form

$$p(n, -n) = \frac{n + 1}{n - 2} p(n - 1, -n + 1).$$

If we repeat this until we hit p(2,-2) we find

$$p(n,-n) = \frac{n+1}{n-2}\frac{n}{n-3}\cdots p(2,-2) = \frac{(n+1)!}{(n-2)!}\frac{1}{2\cdot 3}p(2,-2) = \binom{n+1}{3}p(2,-2).$$

It is convention to normalize $p(2,-2)$ to $\frac{1}{2}$. The reason is to get a better expression in the simplest CFT, the free boson. With this normalization we find

$$p(n,-n) = \frac{1}{12}(n+1)(n-1).$$

In summary we now have found the so called Virasoro algebra with the central charge c given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{12}$$

This exact same procedure can be done with the second copy of Witt algebra.

One may now ask what happened to our understanding of the global generators after the redefinition. Since we know that a finite dimensional algebra does not allow a central extension, we would expect that the cohomology group restricted to those generators vanishes. And indeed, when we redefined the generators, the effect was that $p(n,0) = p(1,-1) = 0$. Therefore the new generators have the same commutation relation as the old ones, without any extension. Therefore they still generate the rotations, dilation, translations and the SCT.

2 Fields and Observables

In the previous chapter we looked at the mathematical structure under which we expect a theory to be symmetric, but we did not mention any details of how such a theory will look like. This is done in the following paragraphs. We will concentrate on the two dimensional case, since a full coverage would surpass the scope of this report and the two dimensional case looks most promising because of its algebra.

We work on the Euclidean two dimensional space $\mathbb{R}^2 \simeq \mathbb{C}$ with the standard identification $z = x^0 + \mathrm{i}x^1$ and $\bar{z} = x^0 - \mathrm{i}x^1$ where we interpret them again as independent. Since we want to have a field theory, we work with fields defined originally on $\mathbb{R}^2$ as $\varphi(x^0, x^1)$. After our complexification we have $\varphi(z,\bar{z})$. We now introduce some terminology for the further discussion.

Definition. *Let $\varphi(z,\bar{z})$ be a field in a CFT.*

- *Fields depending only on z are called chiral fields: $\varphi(z,\bar{z}) = \varphi(z)$*
- *Fields depending only on $\bar{z}$ are called anti-chiral fields: $\varphi(z,\bar{z}) = \varphi(\bar{z})$*
- *A field transforming under dilation as $\varphi'(z,\bar{z}) = \lambda^h\bar{\lambda}^{\bar{h}}\varphi(\lambda z, \bar{\lambda}\bar{z})$ has conformal dimension $(h,\bar{h})$.*
- *A field transforming under local conformal transformations $z' = f(z)$ according to*

  $$\varphi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \varphi(f(z), \bar{f}(\bar{z}))$$

  is called primary field.
- *A field that only transforms under global transformations as a primary field is called quasi primary.*
- *Fields that have a different transformation property are called secondary fields or descendants.*

Before we go ahead and look at the quantization we first have to find the infinitesimal transformations of the fields. We need this result later to proof the so called Conformal Ward Identity.

Let $\varphi(z,\bar{z})$ be a primary field and $f(z) = z + \varepsilon(z)$ a local conformal transformation. We develop both of the derivatives and the new field up to first order of ε.

$$\left(\frac{\partial f}{\partial z}\right)^h = (1 + \partial_z\varepsilon)^h \approx 1 + h\partial_z\varepsilon, \qquad \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} = (1 + \partial_{\bar{z}}\bar{\varepsilon})^{\bar{h}} \approx 1 + \bar{h}\partial_{\bar{z}}\bar{\varepsilon}$$

$$\varphi(z + \varepsilon(z), \bar{z}) \approx \varphi(z) + \varepsilon(z)\partial_z\varphi(z,\bar{z}) \qquad \varphi(z, \bar{z} + \bar{\varepsilon}(\bar{z})) \approx \varphi(\bar{z}) + \bar{\varepsilon}(\bar{z})\partial_{\bar{z}}\varphi(z,\bar{z})$$

Putting all of this together, we find the local transformation of our fields:

$$\delta_{\varepsilon,\bar{\varepsilon}}\varphi = \varphi'(z,\bar{z}) - \varphi(z,\bar{z}) = (h\partial_z\varepsilon + \varepsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}})\varphi(z,\bar{z}). \tag{13}$$

3 Quantization & Radial Ordering

After we studied the primary objects of our theory, we want to quantize this theory. We again work in a two dimensional Euclidean space. As the arbitrary time direction we choose x^0. We compactify the space direction x^1. The reason will be obvious later on. For this we use a circle of radius 1. This means for $z = x^0 + ix^1$ we have $z = z + 2\pi i$. We end up with a cylinder, where the time coordinate is along the cylinder axis and the space component is around the cylinder. Mapping this back to a complex space we can use our previous work, which was all done on $\mathbb{C}$. Such a mapping can be achieved by the coordinate transform $w = exp(z) = exp(x_0) \cdot exp(ix^1)$. Under this transformation the time variable describes the radius of a circle, while the space coordinate is given by an angle in polar coordinates. The notion of time is now equivalent to the notion of radius. This will be used later when we define the time ordering, which in this context could just be called the radial ordering.

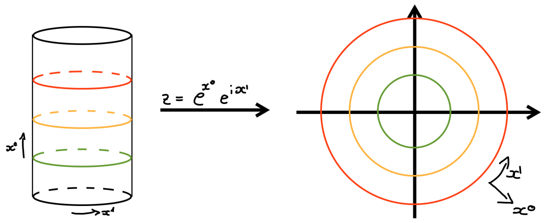

Figure 3: Visual representation of the idea of radial ordering. x^0 is the time which is translated in to the radius.

Former time translations are now mapped to complex dilation and space translations to rotations. Since we showed that the central extension for L_0 and $\bar{L}_0$ vanishes, we can conclude that the time translation generator, or in Quantum Mechanics the Hamiltonian can be written as the dilation operator, given as

$$H = L_0 + \bar{L}_0.$$

3.1 Quantization

We consider a primary field of dimensions $(h, \bar{h})$. Since we expect it to be holomorph we can perform a Laurent expansion around the origin.

$$\varphi(z, \bar{z}) = \sum_{n, \bar{m} \in \mathbb{Z}} z^{-n-h} \bar{z}^{-\bar{m}-\bar{h}} \varphi_{n,\bar{m}}.$$

This can be understood as the equivalent procedure as the quantization using Fourier modes in the normal QFT. In this spirit we now promote the Laurent modes to operators. Using our previous coordinate change, the origin corresponds to $it = x^0 = -\infty$. Therefore we can also define the in-states

$$|\varphi\rangle = \lim_{z, \bar{z} \to 0} \varphi(z, \bar{z})|0\rangle$$

This is again analoge to our procedure in the QFT course. However, similar to the singularity problems in the Witt algebra we need to make sure that the action of the operators in the expansion of $\varphi(z, \bar{z})$ are non singular at $z = 0$. With the exact same calculation as for the Witt algebra we require that

$$\varphi_{n,\bar{m}}|0\rangle = 0 \quad n > -h, \ \bar{m} > -\bar{h}$$

Since all the prefactors z^{-n-h} vanish for $z \to 0$, $n < h$ we can summarize the in states as

$$|\varphi\rangle = \varphi_{-h,-\bar{h}}|0\rangle.$$

We need to find the hermitian conjugate field $\varphi^\dagger$. Since we work with an euclidean metric our time coordinate x^0 is purely imaginary. Therefore, complex conjugation acts as $x^0 \to -x^0$ on this coordinate. In summary, assuming $z = \exp(x^0 + ix^1)$ the hermitian conjugate maps z to $1/\bar{z}$. Having this transformation property of z in mind, we are motivated to define

$$\varphi^\dagger(z, \bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \varphi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right).$$

To find the relation between the daggered components and the original fields, we perform the Laurent expansion of $\varphi^\dagger$.

$$\varphi^\dagger(z,\bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \sum_{n,\bar{m}\in\mathbb{Z}} \bar{z}^{n+h} z^{\bar{m}+\bar{h}} \varphi_{n,\bar{m}} = \sum_{n,\bar{m}\in\mathbb{Z}} \bar{z}^{n-h} z^{\bar{m}-\bar{h}} \varphi_{n,\bar{m}}.$$

But this is the Laurent expansion with shifted indices. Therefore comparing it to the expansion of the first field we find

$$\varphi^\dagger_{n,\bar{m}} = \varphi_{-n,-\bar{m}}.$$

Following the procedure from QFT we define the out states as the daggered in-states. This results in the out state given by

$$\langle\varphi| = \langle 0|\, \varphi_{h,\bar{h}}.$$

3.2 Energy Momentum Tensor

Let's make a short excursion back to a classical CFT. Due to the result of Noether's Theorem we know that any system with a conformal symmetry $x^\mu \mapsto x^\mu + \varepsilon^\mu$ has an associated conserved current $j_\mu + T_{\mu\nu}\varepsilon^\nu$. This tensor is called the energy momentum tensor and is conserved for any conformal mapping ε. We can show that it must be traceless for a classical conformal theory.

$$0 = \partial^\mu j_\mu = (\partial^\mu T_{\mu\nu})\varepsilon^\nu + T_{\mu\nu}\partial^\mu\varepsilon^\nu = 0 + \frac{1}{2}T_{\mu\nu}(\partial^\mu\varepsilon^\nu + \partial^\nu\varepsilon^\mu) = \frac{1}{2}T_{\mu\nu}\eta^{\mu\nu}(\partial\cdot\varepsilon)\frac{2}{d} = \frac{1}{d}T_\mu{}^\mu(\partial\cdot\varepsilon)$$

Since this holds for any (conformal) transformation, we can deduce that the trace of $T_{\mu\nu}$ vanishes. Being focused on two Euclidean dimensions, we want to understand how the energy momentum tensor behaves in this case. We make the change of coordinates as before. Using $x^0 = \frac{1}{2}(z+\bar{z})$ and $x^1 = \frac{1}{2\mathrm{i}}(z-\bar{z})$ with the standard tensor transformation property $T_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}T_{\alpha\beta}$, we find the new transformed quantity

$$\begin{aligned}
T_{zz} &= \frac{1}{4}(T_{00} - 2\mathrm{i}T_{10} - T_{11} = \frac{1}{2}(T_{00} - \mathrm{i}T_{10})\\
T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} + 2\mathrm{i}T_{10} - T_{11}) = \frac{1}{2}(T_{00} + \mathrm{i}T_{10})\\
T_{z\bar{z}} &= T_{\bar{z},z} = T_\mu{}^\mu = 0
\end{aligned}$$

where we used the tracelessness which implies $T_{00} = -T_{11}$. We finally can show that the two non-trivial components can be lifted to a chiral and an anti chiral field. For this we calculate the derivative of T_{zz} with respect to $\bar{z}$ and of $T_{\bar{z}\bar{z}}$ with respect to z. Using the tracelessness of T as well as the symmetry $T_{\mu\nu} = T_{\nu\mu}$ we end up with

$$\begin{aligned}
\partial_{\bar{z}}T_{zz} &= \frac{1}{2}(\partial_0 + \mathrm{i}\partial_1)\frac{1}{2}(T_{00} - \mathrm{i}T_{10}) = \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + \mathrm{i}\partial_1 T_{00} - \mathrm{i}\partial_0 T_{10}) = 0,\\
\partial_z T_{\bar{z}\bar{z}} &= \frac{1}{2}(\partial_0 - \mathrm{i}\partial_1)\frac{1}{2}(T_{00} + \mathrm{i}T_{10}) = \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} - \mathrm{i}\partial_1 T_{00} + \mathrm{i}\partial_1 T_{00}) = 0.
\end{aligned}$$

This shows us that T_{zz} only depends on z, while $T_{\bar{z}\bar{z}}$ is solely a function of $\bar{z}$. But this is just the definition of a chiral or an anti chiral field respectively. We therefore showed that the lifted field belonging to $T_{zz} = T(z)$ is a chiral field, and $T_{\bar{z}\bar{z}} = T(\bar{z})$ is anti chiral.

3.3 Operator Product Expansion

After the discussion of the energy momentum tensor we want to look at the conserved charges connected to it. We know that our conserved current j gives rise to an conserved charge $Q = \int dx^1 j_0$ for any fixed time x^0. Since this conserved charge is associated to the conformal transformation, it is the generator for symmetry transformations. For any operator A we have therefore $\delta A = [Q, A]$, where we need to evaluate both at the same time. Since $x^0 = const$ implies $|z| = const$, the integral over space turns into an integral over a contour C with radius $|z|$. As we have many tools to handle these contour integrals we now justify the choice of cylindrical coordinates. As standard in mathematics and physics, we adopt the convention that the contour integral $\oint dz$ is counter clockwise. Because $T_{00} = T(z) + T(\bar{z})$, the easiest (and therefore hopefully correct) generalization of the conserved charge in these coordinates is

$$Q = \frac{1}{2\pi\mathrm{i}}\oint_C (dz\; T(z)\varepsilon(z) + d\bar{z}\; \bar{T}(\bar{z})\bar{\varepsilon}(\bar{z})).$$

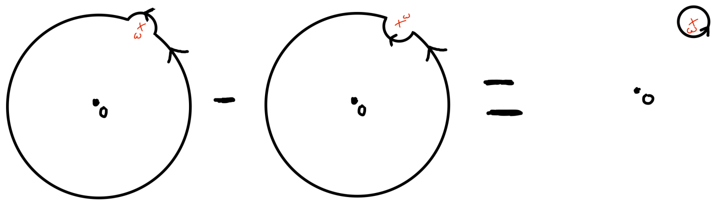

Figure 4: Visualization of the deformation of the integral.

The normalization constant in front of the integral is a convention to get rid of later factors that come in, due to the application of the residue theorem.

In eq. (13) we calculated manually how a field transforms under a conformal transformation. We now have an easier, more straight forward approach using the conserved charge.

$$\delta_{\varepsilon,\bar{\varepsilon}}\varphi(w,\bar{w}) = \frac{1}{2\pi \mathrm{i}}\oint_C dz[T(z)\varepsilon(z),\varphi(w,\bar{w})] + \frac{1}{2\pi \mathrm{i}}\oint_C d\bar{z}[\bar{T}(\bar{z})\bar{\varepsilon}(\bar{z}),\varphi(w,\bar{w})].$$

Before we can compare the two results, we need to differentiate between the case where w and $\bar{w}$ are in- or outside of the contour. This reflects, as mentioned before, the time ordering. Therefore we define the time, or in this case the radial ordering for two operators A and B as

$$\mathcal{R}(A(z)B(w)) = \begin{cases} A(z)B(w) \text{ if } |z| > |w| \\ B(w)A(z) \text{ if } |w| > |z| \end{cases}$$

To make sense, the integral of the commutator needs to be interpreted as

$$\oint_C dz\ [A(z),B(w)] = \oint_{|z|>|w|} dz\ A(z)B(w) - \oint_{|z|<|w|} B(w)A(z),$$

Lets define the infinitesimal circular path around w as $C(w)$. This can be seen in fig. 4.

$$= \oint_{C(w)} \mathcal{R}(A(z)B(w)).$$

Applying this relation to the infinitesimal field transformation we find, if we suppress the anti chiral part,

$$\delta_{\varepsilon,\bar{\varepsilon}}\varphi(w,\bar{w}) = \frac{1}{2\pi \mathrm{i}}\oint_{C(w)} dz\ \mathcal{R}(T(z)\varphi(w,\bar{w}))\varepsilon(z).$$

As we shown in eq. (13), suppressing the anti chiral part we get

$$\delta_{\varepsilon\bar{\varepsilon}}\varphi(w,\bar{w}) = h(\partial_w\varepsilon(w))\varphi(w,\bar{w}) + \varepsilon(w)(\partial_w\varphi(w,\bar{w})).$$

We need to rewrite these expression into integrals. Here, the integralform of the Laurent expansion is used, since it provides the tools to express the derivatives of an holomorphic function as an integral. Applied to this case we have

$$h(\partial_w\varepsilon(w))\varphi(w,\bar{w}) = \frac{1}{2\pi \mathrm{i}}\oint_{C(w)} dz\ h\frac{\varepsilon(z)}{(z-w)^2}\varphi(w,\bar{w}) \tag{14}$$

$$\varepsilon(w)(\partial_w\varphi(w,\bar{w})) = \frac{1}{2\pi \mathrm{i}}\oint_{C(w)} dz\ \frac{\varepsilon(z)}{z-w}\partial_w\varphi(w,\bar{w}) \tag{15}$$

Finally we can compare the two results and can deduce

$$\mathcal{R}(T(z)\varphi(w,\bar{w})) = \frac{h}{(z-w)^2}\varphi(w,\bar{w}) + \frac{1}{z-w}\partial_w\varphi(w,\bar{w}) + f_{non\ singular}$$

where f can be any non singular function. The result is the so called Operator Product Expansion (OPE). That allows to calculate the time ordered product of two operators as a sum of one operator and possibly its derivatives.

It further gives us the option to redefine what we understand under a primary field. We could just use the OPE as the definition of a primary field. This means, every field that has the same OPE with respect to T or $\bar{T}(\bar{z})$ is called a primary field.

From now on, we will suppress the notation of radial ordering and assume that every product of operators is radial ordered or understood as the radial ordered product.

Example. *As an exercise to better understand the OPE, we will calculate the OPE of the energy momentum tensor with itself. This will show that it is not a primary field and is the first instance where we see the central charge popping up. As a starting point we perform a Laurent expansion of the energy momentum tensor. We will shift the indices by 2. The reason for this can be seen in the subsequent calculations.*

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} t_n \qquad t_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z).$$

Lets choose the conformal transformation $\varepsilon(z) = -\varepsilon_n z^{n+1}$*. We now want to calculate the conserved charge of this transformation.*

$$Q_n = \frac{1}{2\pi i}\oint dz - T(z)\varepsilon_n z^{n+1} = -\frac{\varepsilon_n}{2\pi i}\sum_{m\in\mathbb{Z}}\oint dz t_m z^{n-m-1} = -\varepsilon_n t_n$$

This shows that the Laurent modes of the energy momentum tensor are the generators of the infinitesimal conformal transformations which we noted before by L_n*, which is the notation we use from now on again. This is to be expected since the energy momentum tensor, as we already mentioned before is connected to the conserved current of the conformal transformation. As shown before, these generators obey the Virasoro algebra. We can use this to determine the OPE.*

$$\begin{aligned}
0 &= [L_m, L_n] - \left((m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}\right) \\
&= \oint_{C(0)} \frac{dz}{2\pi i}\oint_{C(w)} \frac{dw}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)] - \left(\frac{c}{12}(m^3-m)\delta_{m,-n} + 2(m+1)L_{m+n} - (m+n+2)L_{m+n}\right) \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint_{C(0)} dw\, w^{n+1}\oint_{C(w)} dz\, z^{m+1} T(z)T(w) \\
&\quad - \left(\frac{c}{12}(m^3-m)\delta_{m,-n} + 2(m+1)L_{m+n} + 0 - \frac{1}{2\pi i}\oint dw\,(m+n+2)T(w)w^{m+n+1}\right) \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint_{C(0)} dw\, w^{n+1}\oint_{C(w)} dz\, z^{m+1} T(z)T(w) \\
&\quad - \left(\frac{1}{2\pi i}\oint dw\, \left(\frac{c}{12}(m^3-m)w^{m+n-1} + 2(m+1)w^{m+n+1}T(w) + w^{m+n+2}\partial_w T(w)\right)\right) \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint_{C(0)} dw\, w^{n+1}\oint_{C(w)} dz\, z^{m+1} T(z)T(w) \\
&\quad - \frac{1}{2\pi i}\oint dw\, w^{n+1}\left((m+1)m(m-1)w^{m-2}\frac{c}{2\cdot 2\cdot 3} + 2(m+1)w^m T(w) + w^{m+1}\partial_w T(w)\right) \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint_{C(0)} dw\, w^{n+1}\oint_{C(w)} dz\, z^{m+1} T(z)T(w) \\
&\quad - \frac{1}{2\pi i}\oint dw\, w^{n+1}\frac{1}{2\pi i}\oint dz\, z^{m+1}\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}\right)
\end{aligned}$$

In summary we find the OPE of the energy momentum tensor.

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + f_{non\ singular}$$

We see that it has an extra term in its expansion which is proportional to the central charge c. Therefore, for a non trivial c T is not a primary field.

Example. *If one performs the same calculation as in the previous example, using a primary field, one finds that*

$$[L_m, \varphi_n] = ((h-1)m - n)\varphi_{m+n}. \tag{16}$$

3.4 Two and Three Point Correlators

The goal of any quantum field theory is to derive the correlation functions. The restriction of conformal field theories are strong enough to derive the first two correlators directly. Let's look at the two point correlator $g(z,w) = \langle\varphi_1(z)\varphi_2(w)\rangle$

- Translation invariance: can only depend on the difference between the coordinates $g(z,w) = g(z-w)$
- Rescaling implies $\lambda^{h_1+h_2} g(\lambda(z-w)) = g(z-w)$. Therefore we can directly deduce that $g(z-w) = \frac{d_{12}}{(z-w)^{h_1+h_2}}$

- For the SCT we need a more detailed calculation, since its not trivial to see how g should behave

$$\left\langle \varphi_1(z)\varphi_2(w)\right\rangle \to \left\langle \frac{1}{z^{2h_1}}\frac{1}{w^{2h_2}}\varphi_1(-z^{-1})\varphi_2(-w^{-1})\right\rangle = \frac{1}{z^{2h_1}w^{2h_2}}\frac{d_{12}}{(-z^{-1}+w^{-1})^{h_1+h_2}}$$

  For this to be invariant we need that $h_1 = h_2$.

As a summary we found that all the two point correlators of quasi primary chiral fields can be written as

$$\langle\varphi_i(z)\varphi_j(w)\rangle = \frac{d_{ij}\delta_{h_i,h_j}}{(z-w)^{2h_i}} \tag{17}$$

Example. *Lets use the OPE to determine correlator between the energy momentum tensors.*

$$\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4} + \frac{2\langle T(w)\rangle}{(z-w)^2} + \frac{\langle\partial_w T(w)\rangle}{z-w} = \frac{c/2}{(z-w)^4}$$

Here we used the fact that the exception value $\langle T(w)\rangle$ vanishes. We see that even the non primary field T obeys the above mentioned rule.

For the three point correlator we still are able to pin down the exact form. Let $\langle\varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3)\rangle = g(z_{12}, z_{23}, z_{13})$

- Transnational invarance dictates that $z_{ij} = z_i - z_j$.
- Rescaling again leads to $g(z_{12}, z_{23}, z_{13}) = \frac{C_{123}}{z_{12}^a z_{23}^b z_{13}^c}$ with the constraint that $a + b + c = h_1 + h_2 + h_3$.
- For the SCT we can again perform the same procedure and find the constraints $a = h_1+h_2-h_3$, $b = h_2+h_3-h_1$ and $c = h_1 + h_3 - h_2$.

So we have seen that

$$\langle\varphi_1(z_1)\varphi_2(z_2)\varphi_3(z_3)\rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}}.$$

For higher correlators, there are several possible correlation functions. But we can fix the overall structure in the form of the arguments. We know that they need to be invariant under conformal transformations. Therefore, we develop general functions $\Gamma(x_i)$ that are invariant under all types of conformal transformations. Due to rotations and translations they can again only depend on $|x_i - x_j|$. Scaling-invarance implies that only ratios are allowed in Γ. Having seen how the norm changes under a SCT, we know that

$$|x_i' - x_j'| = \frac{|x_i - x_j|}{\sqrt{(1 - 2f\cdot x_i + f^2x_i^2)(1 - 2f\cdot x_j + f^2x_j^2)}}.$$

Therefore only cross ratios can form invariant coordinates.

Example. *For four points the only possible coordinates for a invariant function are*

$$u = \frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \qquad \textit{and} \qquad v = \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|}$$

By the same arguments as before we find that the four point correlator has the form of

$$\frac{1}{|x_1 - x_2|^{2h}|x_2 - x_3|^{2h}} f(u, v).$$

The more detailed form of the four point correlator needs far more work. For example, it turns out that $f(u, v)$ can't be any function of u and v but has certain restrictions. For example crossing symmetry leads to

$$\left(\frac{v}{u}\right)^h f(u, v) = f(v, u).$$

We won't develop the four point correlator any further, the full derivation is shown in [3].

3.5 Ward Identities

The Ward identities are connected to the symmetry of the action. We first derive them in a general QFT context and then we investigate how they can be formulated in a CFT. Let's write a general infinitesimal transformation as $\Phi'(x) = \Phi(x) - \mathrm{i}\omega_a G_a \Phi(x)$ where ω_a is a collection of infinitesimal parameters. We assume that the action is invariant under such a transformation, as long as the parameters are constant. We now use the path integral formalism and interpret the parameters not as constants but as functions on the space. The action is not invariant anymore and we can perform a change of variables to Φ' in the path integral. Let $X = \Phi(x_1)\cdots\Phi(x_n)$. If we assume that the functional integration measure stays invariant and develop the path integral up to the first order in w_a we find

$$\langle X\rangle = \frac{1}{Z}\int d\Phi'(X+\delta X)\exp\Big\{-(S[\Phi]+\int dx\partial_\mu J^\mu_a\omega_a(x))\Big\},$$

where δX are all the first order terms of the form $\Phi(x_1)\cdots\Phi(x_j)(\mathrm{i}\omega_a(x)G_a\Phi(x_{j+1}))\cdots\Phi(x_n)$. The zero order terms match and therefore we end up with the following expression:

$$\langle\delta X\rangle = \int dx\partial_\mu\langle j^\mu_a(x)X\rangle\omega_a(x)$$

This can be easily rewritten if we introduce a delta function as

$$\partial_\mu\langle j^\mu_a(x)\Phi(x_1)\cdots\Phi(x_n)\rangle = -\mathrm{i}\sum_{j=1}^{n}\delta(x-x_j)\langle\Phi(x_1)\cdots G_a\Phi(x_j)\cdots\Phi(x_n)\rangle$$

We actually could use this as a quantization procedure using path integrals. But we decided to use the Laurent expansion for the quantization instead, since it is still possible for simple CFTs. We now can apply this to the two dimensional case. Let φ be primary.

$$\begin{aligned}\Big\langle\oint\frac{dz}{2\pi\mathrm{i}}\varepsilon(z)T(z)\varphi_1(w_1,\bar{w}_1)\cdots\varphi_n(w_n,\bar{w}_n)\Big\rangle\\ &= \sum_{i=1}^{n}\Big\langle\varphi_1(w_1,\bar{w}_1)\cdots\Big(\oint_{w_i}\frac{dz}{2\pi\mathrm{i}}\varepsilon(z)T(z)\varphi_i(w_i\bar{w}_i)\Big)\cdots\varphi_n(w_n,\bar{w}_n)\Big\rangle\\ &= \sum_{i=1}^{n}\Big\langle\varphi_1(w_1,\bar{w}_1)\cdots\Big(h_1\partial\varepsilon(w_i)+\varepsilon(w_i)\partial_{w_i}\Big)\varphi_i(w_i,\bar{w}_i)\cdots\varphi_n(w_n,\bar{w}_n)\Big\rangle\end{aligned}$$

If we use eq. (14), we find

$$0 = \oint\frac{dz}{2\pi\mathrm{i}}\varepsilon(z)\Bigg[\Big\langle T(z)\varphi_1(w_1,\bar{w}_1)\cdots\varphi_n(w_n,\bar{w}_n)\Big\rangle - \sum_{i=1}^{n}\Big(\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\partial_{w_i}\Big)\Big\langle\varphi_1(w_1,\bar{w}_1)\cdots\varphi_n(w_n,\bar{w}_n\Big\rangle\Bigg]$$

Since this must hold for all holomorphic $\varepsilon(z)$, the integrand has to vanish. This leads us to the so called Ward identity for conformal field theory

$$\Big\langle T(z)\varphi_1(w_1,\bar{w}_1)\cdots\varphi_n(w_n,\bar{w}_n)\Big\rangle = \sum_{i=1}^{n}\Big(\frac{h_i}{(z-w_i)^2}+\frac{1}{z-w_i}\partial_{w_i}\Big)\Big\langle\varphi_1(w_1,\bar{w}_1)\cdots\varphi_n(w_n,\bar{w}_n)\Big\rangle \tag{18}$$

3.6 General OPE

We can use the form of the two and three point function as a motivation to determine the general form of the product of two quasi primary field. We make the Ansatz

$$\varphi_i(z)\varphi_j(w) = \sum_{k,n\geq 0}C^k_{ij}\frac{a^n_{ijk}}{n!}\frac{1}{(z-w)^{(h_i+h_j-h_k-n)}}\partial^n\varphi_k(w)$$

What is the exact reason behind this Ansatz? As we know from the previous examples, due to the scaling behavior under dilation we need that the denominator is of form $(z-w)$. Further we have chosen two different variables, where C encodes information about the used fields and a about the derivative. Let's use this Ansatz together with the two point correlation function.

$$\begin{aligned}
\Big\langle (\varphi_i(z)\varphi_j(1))\varphi_k(0)\Big\rangle &= \sum_{l,n\geq 0} C_{ij}^l \frac{a_{ijl}^n}{n!} \frac{\langle \partial^n \varphi_l(1)\varphi_k(0)\rangle}{(z-1)^{h_i+h_j-h_l-n}} \\
&= \sum_{l,n\geq 0} C_{ij}^l \frac{a_{ijl}^n}{n!} \frac{\langle \partial_z^n \varphi_l(z)\varphi_k(0)\rangle|_{z=1}}{(z-1)^{h_i+h_j-h_l-n}} \\
&= \sum_{l,n\geq 0} C_{ij}^l \frac{a_{ijl}^n}{n!} \frac{1}{(z-1)^{h_i+h_j-h_l-n}} \partial_z^n \left(\frac{d_{lk}\delta_{h_l,h_k}}{z^{2h_k}}\right)\Bigg|_{z=1} \\
&= \sum_{l,n\geq 0} C_{ij}^l \frac{a_{ijl}^n}{n!} \frac{1}{(z-1)^{h_i+h_j-h_l-n}} (-1)^n n! \delta_{h_k,h_l} \binom{2h_k+n-1}{n} \\
&= \sum_{l,n\geq 0} C_{ij}^l d_{lk} a_{ajk}^n \binom{2h_k+n-1}{n} \frac{(-1)^n}{(z-1)^{h_i+h_j-h_l-n}} .
\end{aligned}$$

However this should be of the same form as the three point correlator. Using this, we can conclude

$$\sum_{l,n\geq 0} C_{ij}^l d_{lk} a_{ajk}^n \binom{2h_k+n-1}{n} \frac{(-1)^n}{(z-1)^{h_i+h_j-h_l-n}} = \frac{C_{ijk}}{(z-1)^{h_i+h_j-h_k} z^{h_i+h_k-h_j}} .$$

Multiplying both sides with $(z-1)^{h_i+h_j-h_k}$ we find

$$\sum_{l,n\geq 0} C_{ij}^l d_{lk} a_{ajk}^n \binom{2h_k+n-1}{n} (-1)^n (z-1)^n = \frac{C_{ijk}}{(1+z-1)^{h_i+h_k-h_j}} .$$

We can use the full Taylor expansion

$$\frac{1}{(1+x)^m} = \sum_{n\in\mathbb{N}} (-1)^n \binom{m+n-1}{n} x^n$$

to connect the individual coefficients. Let $x = z-1$. Now, comparing coefficients we find

$$a_{ijk}^n = \binom{2h_k+n-1}{n}^{-1} \cdot \binom{h_k+h_i-h_j+n-1}{n} \quad , \quad C_{ijk} = C_{ij}^l \cdot d_{lk} \tag{19}$$

This gives the complete OPE for two chiral primary fields.

While discussing the OPE of the energy momentum tensor we have seen that the OPE is closely connected to the commutator. We can perform the same calcualtion as we did with the energy momentum tensor using the expression

$$\varphi_i(z) = \sum_m \varphi_{m,(i)} z^{-m-h_i}$$

to find the commutation relation

$$[\varphi_{m,(i)}, \varphi_{n,(j)}] = \sum_k C_{ij}^k p_{ijk}(m,n) \varphi_{m+n,(k)} + d_{ij}\delta_{m,-n} \binom{m+h_i-1}{2h_i-1}$$

where we have

$$p_{ijk}(m,n) = \sum_{r,s\in\mathbb{Z}_0^+} C_{r,s}^{ijk} \binom{-m+h_i-1}{r} \cdot \binom{-n+h_j-1}{s} \tag{20}$$

$$C_{r,s}^{ijk} = (-1)^r \frac{(2h_k-1)!}{(h_i+h_j+h_k-2)!} \prod_{t=0}^{s-1} (2h_i-2-r-t) \prod_{u=0}^{r-1} (2h_j-2-s-u) \tag{21}$$

This calculation is very long and therefore not written here. The idea is the same as in the proof of the OPE of the energy momentum tensor. The first step is to use the expansion of the fields and then assume eq. (20). Write the commutator in an integral just as we did with the energy momentum tensor and use the assumption. Performing a few algebraic steps one recovers the OPE conditions eq. (19). This now can be rewritten as a formal proof of eq. (20) without assuming the result but just assume what we already have proven in eq. (19).

So we calculated the commutators for conformal quasi primary fields. But as mentioned before, there are many fields that are not quasi primary. Therefore, we have not specified the commutations for all possible fields!

Example. *Let's compute the norm of the state $\varphi_{-n,(i)}$. Let's assume $n \geq h$. So we find*

$$\begin{aligned}||\varphi_{-n,(i)}|0\rangle||^2 &= \langle 0|\varphi_{n,(i)}\varphi_{-n,(i)}|0\rangle \\ &= \langle 0|[\varphi_{n,(i)}, \varphi_{-n,(i)}]|0\rangle \\ &= C_{ii}^{j} p_{iij}(n,-n)\langle 0|\varphi_{0,(j)}|0\rangle + d_{ii}\binom{n+h_i-1}{2h_i-1} \\ &= d_{ii}\binom{n+h_i-1}{2h_i-1}\end{aligned}$$

Now we can choose $n = h$ and find that the norm of a state $|\varphi\rangle$ is given by the structure constant of the two point correlator,

$$\langle\varphi|\varphi\rangle = d_{\varphi\varphi}.$$

3.7 Current Algebras

A chiral field with conformal dimension of 1 is usually called a current. Those have a special algebra connected to them, which is called the Kač-Moody algebra or current algebra. To determine it, we express the currents as a Laurent expansion and assume that we have N such currents given in our theory. Using what we derived in the section about the general OPE, we find

$$\left[j_{(i)m}, j_{(j)n}\right] = \sum_k C_{ij}^k p_{111}(m,n) j_{(k),m+n} + d_{ij} m \delta_{m,-n}. \tag{22}$$

We can use the expression from the previous section for $p_{111}(m,n)$ to find that it equals to 1. Since the commutator is antisymmetric, we conclude that the coefficients C_{ij}^k are antisymmetric in the lower indices.

We now perform a rotation among the fields such that the matrix d_{ij} is diagonal. If we rescale all the fields by certain factors, we can achieve that the matrix $d'_{ij} \propto \delta_{ij}$. In this new basis we can denote the structure constants as f^{ijl} and are now able to write

$$\left[j_m, j_n^j\right] = \sum_l f^{ijl} j_{m+n}^l + km\delta^{ij}\delta_{m,-n}.$$

This algebra is usually called the Kač-Moody algebra.

3.8 Normal Ordering

In this part we will introduce the normal ordering of operators as we know it from QFT. For that we first need to identify what we will consider as a creation and what as an annihilation operator. Remembering that $\varphi_{n,\bar{m}}|0\rangle = 0$ if $n > -h, \bar{m} > -\bar{h}$, we can interpret these operators as annihilation operators. We have seen earlier that the Hamiltonian can be expressed as $H = L_0 + \bar{L}_0$. Therefore we interpret the eigenvalues of L_0 as the chiral energy. We require $L_0|0\rangle = 0$. For a chiral primary we can calculate

$$L_0\varphi_n|0\rangle = [L_0, \varphi_0]|0\rangle = -n\varphi_n|0\rangle$$

Putting those two things together, we see that the chiral energy is bounded from below with the values (h+m) for $m \geq 0$. Since we want our creation operators to create states with positive energy, we conclude that

φ_n where $n > -h$ are annihilation operators,

φ_n where $n \leq -h$ are creation operators.

We can do the exact same for the anti chiral sector using $\bar{L}_0$. The normal ordering prescription is to put all creation operators to the left. If we now look back to the OPE, the regular part naturally needs to be normal ordered, and therefore we have

$$\varphi(z)\chi(w) = f_{\text{singular}} + \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} \mathcal{N}(\chi\partial^n\varphi)(w)$$

where we use the operator $\mathcal{N}$ for the normal ordering. If we apply $\frac{1}{2\pi i}\oint dz(z-w)^{-1}$ to both sides of the equations, this takes out the $n = 0$ term on the RHS. We therefore get

$$\mathcal{N}(\chi\varphi)(w) = \oint_{C(w)} \frac{dz}{2\pi i} \frac{\varphi(z)\chi(w)}{z-w}. \tag{23}$$

Performing a Laurent expansion of the LHS, the coefficients are given by

$$\mathcal{N}(\chi\varphi)_n = \oint_{C(0)} \frac{dw}{2\pi i} w^{n+h^{\varphi}+h^{\chi}-1} \mathcal{N}(\chi\varphi)(w)$$

where we made the usual shift of the Laurent expansion

$$\mathcal{N}(\chi\varphi)(w) = \sum_{\mathbb{Z}} w^{-n-h^{\varphi}-h^{\chi}} \mathcal{N}(\chi\varphi)_n.$$

If we now plug in the relation 23 we find

$$\begin{aligned}\mathcal{N}(\chi\varphi)_n &= \oint_{C(0)} \frac{dw}{2\pi i} w^{n+h^{\varphi}+h^{\chi}-1} \oint_{C(w)} \frac{dz}{2\pi i} \frac{\varphi(z)\chi(w)}{z-w} \\ &= \oint \frac{dw}{2\pi i} w^{n+h^{\varphi}+h^{\chi}-1} \left(\oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{\varphi(z)\chi(w)}{z-w} - \oint_{|z|<|w|} \frac{dz}{2\pi i} \frac{\chi(w)\varphi(z)}{z-w} \right)\end{aligned}$$

We now concentrate on the first term. For that we first express φ and χ as a Laurent series. Using the geometric series and $\frac{1}{z-w} = \frac{1}{z(1-w/z)}$ we find

$$\begin{aligned}\oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{\varphi(z)\chi(w)}{z-w} &= \oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{1}{z-w} \sum_{r,s} z^{-r-h^{\varphi}} w^{-s-h^{\chi}} \varphi_r \chi_s \\ &= \oint_{|z|>|w|} \frac{dz}{2\pi i} \sum_{p\geq p} \sum_{r,s} z^{-r-h^{\varphi}-p-1} w^{-s-h^{\chi}+p} \varphi_r \chi_s\end{aligned}$$

Only the z^{-1} term contributes, so we find a δ function. This ses $r = -h^{\varphi} - p$. If we now take the outer integral into account, we find that the first term is given by

$$\oint \frac{dw}{2\pi i} \sum_{p\geq 0} \sum_{s} w^{-s-h^{\chi}+p+n+h^{\varphi}+h^{\chi}-1} \varphi_{-h^{\varphi}-p} \chi_s.$$

Again only the w^{-1} term contributes and we therefore have that the first term is given by

$$\sum_{k\leq -h^{\varphi}} \varphi_k \chi_{n-k}.$$

If we perform the same calculation for the second term we finally find the expression for the Laurent modes of a normal ordered product as

$$\mathcal{N}(\chi\varphi)_n = \sum_{k>-h^{\varphi}} \chi_{n-k}\varphi_k + \sum_{k\leq -h^{\varphi}} \varphi_k \chi_{n-k}. \tag{24}$$

4 The Free Boson

To find a non trivial example for a conformal field theory, we look at string theory. We will work out the central charge of a free Boson. We won't go into detail how to derive the world sheet action or the general energy momentum tensor for such a theory, since this would need another project report. Here we will just state these quantities as an assumption. Details can be found in [2].

4.1 Conformal Symmetry

We define a real massless scalar field $X(x^0, x^1)$ on a cylindrical euclidean space with the identification $x^1 = x^1 + 2\pi$. The euclidean world sheet action is given by

$$\mathcal{S} = \frac{1}{4\pi\kappa} \int dx^0 dx^1 \left((\partial_0 X)^2 + (\partial_1 X)^2 \right).$$

Why does this action lead to a conformal invariant theory? The short answer is that we lack any scale factor. The long answer: It doesn't. Or more precisely, it only does for a field with vanishing conformal dimension. We can see this after we make a coordinate transform onto the complex plane using $z = e^{x^0} e^{ix^1}$. Under this transformation the action now reads

$$\mathcal{S} = \frac{1}{4\pi\kappa} \int dz d\bar{z}\, \partial X \cdot \bar{\partial} X.$$

Let us now verify that for a field of vanishing conformal dimension ($X'(x,\bar{z}) = X(w,\bar{w})$) the action stays invariant.

$$\begin{aligned}
\mathcal{S}' &= \frac{1}{4\pi\kappa}\int dzd\bar{z}\partial_z X'(x,\bar{z})\cdot\partial_{\bar{z}}X'(z,\bar{z}) \\
&= \frac{1}{4\pi\kappa}\int \frac{\partial z}{\partial w}dw\frac{\partial\bar{z}}{\partial\bar{w}}d\bar{w}\frac{\partial w}{\partial z}\partial_w X(w,\bar{w})\cdot\frac{\partial\bar{w}}{\partial\bar{z}}\partial_{\bar{w}}X(w,\bar{w}) \\
&= \frac{1}{4\pi\kappa}\int dwd\bar{w}\partial_w X(w,\bar{w})\cdot\partial_{\bar{w}}X(w,\bar{w}) \\
&= \mathcal{S}.
\end{aligned}$$

Indeed, for a field X of conformal dimension (0,0) this action is conformal invariant. We now derive the equation of motion for this action.

$$\begin{aligned}
0 &= \delta_X\mathcal{S} \\
&= \frac{1}{4\pi\kappa}\int dzd\bar{z}\left(\partial\delta X\cdot\bar{\partial}X + \partial X\cdot\bar{\partial}\delta X\right) \\
&= \frac{1}{4\pi\kappa}\int dzd\bar{z}\left(\partial(\delta X\cdot\bar{\partial}X) - \delta X\cdot\partial\bar{\partial}X + \bar{\partial}(\partial X\cdot\delta X) - \bar{\partial}\partial X\cdot\delta X\right) \\
&= -\frac{1}{2\pi\kappa}\int dzd\bar{z}\delta X(\partial\bar{\partial}X)
\end{aligned}$$

Therefore the equation of motion is given by $\partial\bar{\partial}X(z,\bar{z}) = 0$. This shows that this theory has a chiral and an anti chiral field.

$$j(z) = \mathrm{i}\partial X(z,\bar{z}) \qquad \text{and} \qquad \bar{j}(\bar{z}) = \mathrm{i}\bar{\partial}X(z,\bar{z})$$

Let's reconsider the derivation of the equation of motion. Here we already used these fields indirectly. The interesting phenomena is that they are primary and we see that they have conformal dimension (1,0) and (0,1) respectively. It turns out that the fields j are more useful than the original fields X, which will be even more obvious after we calculated the correlators.

4.2 Correlator

Lets define $K(z,\bar{z},w\bar{w}) = \langle X(z,\bar{z})X(w,\bar{w})\langle$. We know that this correlator is also the propagator which has to satisfy the equations of motion as a Greens function.

$$\partial_z\partial_{\bar{z}}K = -2\pi\kappa\delta^{(2)}(z-w)$$

From complex analysis we already know this equation and its solution as

$$K(z,\bar{z},w,\bar{w}) = -\kappa\log\left(|z-w|^2\right).$$

This equation seems rather odd for several reasons. First it shows that the field X is not quasi primary and secondly the correlator increases with greater separation of the two coordinates. The chiral fields on the other hand are primary fields and therefore the correlator should be of the previously found form eq. (17). To see that this holds, we calculate

$$\mathrm{i}^2\langle\partial_z X(z,\bar{z})\partial_w X(w,\bar{w})\rangle = \kappa\partial_z\partial_w\left(\log(z-w)+\log(\bar{z}-\bar{w})\right) = -\frac{\kappa}{(z-w)^2}.$$

In summary we found that the chiral fields indeed are primary. Doing the same calculation for $\bar{j}$, one gets the same correlator with barred coordinates. Therefore we have the normalization constant of the two point function given as $d_{jj} = \kappa$. Since we only have one chiral current in our theory, the anti symmetry of the Laurent Modes in eq. (22) imply that we have the commutator

$$[j_m, j_n] = \kappa m\delta_{m+n,0}.$$

We now can derive the energy momentum tensor for this theory, where we use the fact that the energy momentum tensor in String theory can be derived by differentiating the action with respect to the metric, we allow for a further normalization constant γ.

$$T_{ab} = 4\pi\kappa\gamma\frac{1}{\sqrt{|g|}}\frac{\delta\mathcal{S}}{\delta g^{ab}}$$

This results in

$$T_{zz} = \gamma \partial X \partial X = \gamma jj, \qquad T_{z,\bar{z}} = T_{\bar{z},z} = 0, \qquad T_{\bar{z}\bar{z}} = \gamma \bar{\partial} X \bar{\partial} X.$$

Now we need to take the normal ordered product, which results in the expression

$$T(z) = \gamma \mathcal{N}(jj)(z)$$

The normalization constant which is still left can be fixed by our requirement that $j(z)$ is primary with dimension $h = 1$. We expand $T(z)$ in Laurent modes which gives

$$L_n = \gamma \mathcal{N}(jj)_n = \gamma \sum_{k>-1} j_{n-k} j_k + \gamma \sum_{k\leq -1} j_k j_{n-k},$$

where we used eq. (24). Calculating the following commutator will fix the constant γ.

$$\begin{aligned}
[L_m, j_n] &= \gamma [\mathcal{N}(jj)_m, j_n] \\
&= \gamma \sum_{k>-1} \left(j_{m-k} \left[j_k, j_n \right] + \left[j_{m-k}, j_n \right] j_k \right) + \gamma \sum_{k\leq -1} \left(j_k \left[j_{m-k}, j_n \right] + \left[j_k, j_n \right] j_{m-k} \right) \\
&= -2\gamma\kappa n j_{m+n}
\end{aligned}$$

If we compare that to eq. (16), we can conclude that $2\gamma\kappa = 1$. As a result, we now know that the chiral part of the energy momentum tensor is given by

$$T(z) = \frac{1}{2\kappa} \mathcal{N}(jj)(z)$$

4.3 Central Charge

We now have everything to finally determine the central charge of the free boson. To do so we use the generator L_2.

$$\langle 0 | L_2 L_{-2} | 0 \rangle = \langle 0 | \left[L_2, L + -2 \right] | 0 \rangle = \frac{c}{2}$$

Lets recall that the Laurent modes are just the Laurent coefficients for the energy momentum tensor, therefore we have $L_{\pm 2} = \frac{1}{2\kappa} \mathcal{N}(jj)_{\pm 2}$. Using this we rewrite

$$\begin{aligned}
L_{-2} | 0 \rangle &= \frac{1}{2\kappa} j_{-1} j_{-1} | 0 \rangle \\
\langle 0 | L_2 &= \frac{1}{2\kappa} \langle 0 | j_1 j_1
\end{aligned}$$

Plugging this into the commutator we get

$$\begin{aligned}
\frac{c}{2} &= \frac{1}{\kappa^2} \langle 0 | j_1 j_1 j_{-1} j_{-1} | 0 \rangle \\
&= \frac{1}{4\kappa^2} \left(\langle 0 | j_1 j_{-1} j_1 j_{-1} | 0 \rangle + \langle 0 | j_1 \left[j_1, j_{-1} \right] j_{-1} \right) \\
&= \frac{1}{4\kappa^2} \left(\langle 0 | \left[j_1, j_{-1} \right] \left[j_1, j_{-1} \right] | 0 \rangle + \kappa \langle 0 | \left[j_1, j_{-1} \right] | 0 \rangle \right) \\
&= \frac{1}{4\kappa^2} 2\kappa^2 = \frac{1}{2}
\end{aligned}$$

Therefore we find that the central charge for the free boson on a 2 dimensional worldsheet is given by $c = 1$.

5 Other Conformal Field Theories

We have seen that conformal invariance implies strong constraints, so strong that we can actually write out many of the generators just by the knowledge that the theory is conformal invariant. We also have seen that a certain simple string theory can be formulated as a conformal field theory. But why should we study a theory that only applies to very trivial theories? It turns out that we are not restricted to only those. There are many non trivial systems obeying a conformal symmetry.

- Statistical Mechanics
  - Ising model on a hypercubic lattice near its critical point
  - Critical Potts model as a generalization of the Ising model in various dimensions
  - Critical O(N) models as generalization of the Potts model
- Theories build from generalized free fields (Mean Field Theories)
- Large N limit for the Yang - Mills Theories of type SU(N)
- String Theories
- Scale invariant theories (with certain restrictions, for example d=2)
  - High energy effective theory ($m \approx 0$)

Further reasons to study this topic are results like the AdS/CFT correspondence. It is a duality between certain string theories on an Anti-de-Sitter space which describe a possible formulation of a quantum gravity and certain conformal field theories formulated on the boundary of this space. This serves as a possible stepping stone for a unification of quantum gravity and particle physics. This duality is not yet fully proven.

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